Test Code: PCB (short answer type) 2016

M.Tech. in Computer Science

Syllabus and Sample Questions

The selection test for M.Tech. in Computer Science will consist of two parts.

- Test **MMA** (objective type) in the forenoon session is the 1^{st} part, and
- Test **PCB** (short answer type) in the afternoon session is the 2^{nd} part.

  The **PCB** test will consist of two groups.

  ◇ **Group A** (30 Marks) : All candidates have to answer questions on analytical ability and mathematics at the undergraduate level.

  ◇ **Group B** (70 Marks) : A candidate has *to choose exactly one of the following five sections*, from which questions have to be answered:
  (i) Mathematics, (ii) Statistics, (iii) Physics, (iv) Computer Science, and (v) Engineering and Technology.
  While questions in the first three sections will be at postgraduate level, those for the last two sections will be at B.Tech. level.

The syllabus and sample questions for the **MMA** test are available separately. The syllabus and sample questions for the **PCB** test are given below.

Note:

1. Not all questions in this sample set are of equal difficulty. They may not carry equal marks in the test. More sample questions are available on the website for M.Tech(CS) at *http://www.isical.ac.in/~deanweb/MTECHCSSQ.html*
2. Each of the two tests **MMA** and **PCB**, will have individual qualifying marks.

SYLLABUS for Test PCB

Group A

Elementary Euclidean geometry and trigonometry.
Elements of set theory, Functions and relations, Permutations and combinations, Principle of inclusion and exclusion, Pigeon-hole principle.
Theory of equations, Inequalities.
Elementary number theory, divisibility, congruences, primality.
Determinants, matrices, solutions of linear equations, vector spaces, linear independence, dimension, rank and inverse.
Limits, continuity, sequences and series, differentiation and integration with applications, maxima-minima.

Group B

Mathematics

In addition to the syllabus for Mathematics in Group A, the syllabus includes:

Linear algebra – vector spaces and linear transformations, direct sum, matrices and systems of linear equations, characteristic roots and characteristic vectors, Cayley-Hamilton theorem, diagonalization and triangular forms, quadratic forms.

Abstract algebra –
Groups: subgroups, products, cosets, Lagranges theorem, group homomorphism, normal subgroups and quotient groups, permutation groups, Sylow theorems.
Rings and integral domains: subrings, ring homomorphism, ideals and quotient rings, prime and maximal ideals, products, Chinese remainder theorem, prime and irreducible elements, principal ideal domain, unique factorization domains.
Polynomial rings: division algorithm, roots of polynomials.
Fields: characteristic of a field, field extensions, finite fields.

Calculus and real analysis – real numbers, limits, continuity, uniform continuity of functions, differentiability of functions of one or more variables and applications, convergence of sequences and series; indefinite integral, fundamental theorem of Calculus, Riemann integration, improper integrals, double and multiple integrals and applications, sequences and series of functions, uniform convergence, solutions of ordinary differential equations.

Graph Theory – connectedness, trees, vertex coloring, planar graphs, Eulerian graphs, Hamiltonian graphs, digraphs and tournaments.

Statistics

Notions of sample space and probability, combinatorial probability, conditional probability, Bayes' theorem and independence.
Random variable and expectation, moments, standard univariate discrete and continuous distributions, sampling distribution of statistics based on normal samples, central limit theorem, approximation of binomial to normal, Poisson law.
Multinomial, bivariate normal and multivariate normal distributions.
Descriptive statistical measures, product-moment correlation, partial and multiple correlation.
Regression – simple and multiple.
Elementary theory and methods of estimation – unbiasedness, minimum variance, sufficiency, maximum likelihood method, method of moments, least squares methods.
Tests of hypotheses – basic concepts and simple applications of Neyman-Pearson lemma, confidence intervals.
Tests of regression, elements of non-parametric inference, contingency tables and Chi-square, ANOVA, basic designs (CRD/RBD/LSD) and their analyses, elements of factorial designs.
Conventional sampling techniques, ratio and regression methods of estimation.

Physics

Classical mechanics – Lagrangian and Hamiltonian formulation, symmetries and conservation laws, motion in central field of force, planetary motion, simple harmonic motion - damped, undamped and forced, special theory of relativity.
Electrodynamics – electrostatics, magnetostatics, electromagnetic induction, self and mutual inductance, capacitance, Maxwell's equation in free space.
Nonrelativistic quantum mechanics – Planck's law, photoelectric effect, Compton effect, wave-particle duality, Heisenberg's uncertainty principle, Schrodinger equation and applications.
Thermodynamics and statistical Physics – laws of thermodynamics and their consequences, thermodynamic potentials and Maxwell's relations, chemical potential, phase equilibrium, phase space, microstates and macrostates, partition function, free energy, classical statistics.
Atomic and molecular physics – quantum states of an electron in an atom, Hydrogen atom spectrum, electron spin, spin-orbit coupling, fine structure, Zeeman effect.
Condensed matter physics – crystal classes, 2D and 3D lattice, reciprocal lattice, bonding, diffraction and structure factor, point defects and dislocations, lattice vibration, free electron theory, electron motion in periodic potential, energy bands in metals, insulators and semiconductors.
Basic nuclear physics – nuclear properties, nuclear forces, nuclear structures, nu-

clear reactions, radioactive nuclear decay.
Electronics – semiconductor physics; diodes - clipping, clamping, rectification; Zener regulated power supply, bipolar junction transistor - CC, CB, and CE configurations; transistor as a switch; amplifiers.
Operational Amplifier and its applications – inverting & noninverting amplifiers, adder, integrator, differentiator, waveform generator, comparator, Schmidt trigger.
Digital integrated circuits – NAND and NOR gates as building blocks, XOR gates, half and full adder.

Computer Science

Data structures - array, stack, queue, linked list, binary tree, heap, AVL tree, B-tree.
Discrete Mathematics - recurrence relations, generating functions, graph theory - paths and cycles, connected components, trees, digraphs.
Programming languages - Fundamental concepts - abstract data types, procedure call and parameter passing, dynamic memory allocation, C and C++.
Design and analysis of algorithms - Asymptotic notation, searching, sorting, selection, graph traversal, minimum spanning tree.
Switching Theory and Logic Design - Boolean algebra, minimization of Boolean functions, combinational and sequential circuits - synthesis and design.
Computer organization and architecture - Number representation, computer arithmetic, memory organization, I/O organization, microprogramming, pipelining, instruction level parallelism.
Operating systems - Memory management, processor management, critical section problem, deadlocks, device management, file systems.
Formal languages and automata theory - Finite automata and regular expressions, pushdown automata, context-free grammars, Turing machines, elements of undecidability.
Database management systems - Relational model, relational algebra, relational calculus, functional dependency, normalization (up to 3-rd normal form).
Computer networks - OSI, LAN technology - Bus/tree, Ring, Star; MAC protocols; WAN technology - circuit switching, packet switching; data communications - data encoding, routing, flow control, error detection/correction, Inter-networking, TCP/IP networking including IPv4.

Engineering and Technology

C Programming language.
Gravitation, moments of inertia, particle dynamics, elasticity, friction, strength of materials, surface tension and viscosity.
Laws of thermodynamics and heat engines.
Electrostatics, magnetostatics and electromagnetic induction.
Laws of electrical circuits – transient and steady state responses of resistive and

reactive circuits.
D.C. generators, D.C. motors, induction motors, alternators, transformers.
Diode circuits, bipolar junction transistors & FET devices and circuits, oscillator, operational amplifier.
Boolean algebra, Minimization of Boolean functions.
Combinatorial and sequential circuits – multiplexer, de-multiplexer, encoder, decoder, flip-flops, registers and counters, A/D and D/A converters.

SAMPLE QUESTIONS

Group A

A1. How many positive integers less than or equal to 1000, are not divisible by none of 7, 11, and 13?

A2. Find all real solutions of the equation $x^2 - |x-1| - 3 = 0$.

A3. For a prime p and a positive integer n, we define

$$A_{p,n} = \{(x,r) : 1 \leq x \leq n, r \text{ is a positive integer, } p^r \text{ divides } x\}.$$

Describe the set $A_{p,n}$ for $p = 5$, $n = 100$.

A4. A 3×3 magic square is an arrangement of the numbers from a set of odd integers $\{1, 3, 5, \ldots, 17\}$ in a 3×3 square grid, where the numbers in each row, in each column, and in the main and secondary diagonals, all add up to 27. Prove that the element at the center of the grid is 9.

Group B

Section I : Computer Science

C1. An integer a belonging to an array A of n integers is said to be the *majority* element if it appears more than $\lfloor n/2 \rfloor$ times in A.

Note that if two different elements in A are removed, then the majority element of A, if it exists, remains unchanged. This idea can be turned into a linear time recursive algorithm as follows.

Let $c = A[0]$ (array indexing is as in `C` programming language). We scan A from $A[1]$ onwards; a counter, initially set to 1, is incremented if the current element is same as c, the counter is decremented if the current element is different from c. If the counter becomes zero at any point, we recurse on the rest of A; else if we finish scanning A, we have a possible majority element. We compute its frequency to confirm if it is the majority element.

Given below is the sketch of a `C` program. Copy the entire lines with blank(s) (along with commented line numbers) and fill them properly in your answer sheet.

```
#include<stdio.h>
#include<stdlib.h>
#include<math.h>

int candidate(int m, int *array, int n)
{
  int j, c, counter=1;
  j = m; c = array[m];
/*line 1*/ while( ______ && ______ ){
     j = j+1;
/*line 2*/ if(array[j]== ______) counter++;
     else counter--;
  }
/*line 3*/ if(j == ______) return c;
/*line 4*/ else return candidate(___, ___, ___);
}

void main(void)
{
  int n, *array, i, c, count=0;

  printf("\n Size of the array::> ");
```

```
    scanf("%d", &n);

    array = (int *)calloc(n,sizeof(int));
    if(array == NULL)
     {
       printf("\n No space! \n ");
       exit(0);
     }
    printf("\n Input the array::>");
    for(i=0;i<n;i++)
/*line 5*/ scanf("%d",______);

/*line 6*/ c = candidate(0,______,n);
    for(i=0;i<n;i++){
/*line 7*/ if(array[i]== ______) count=count+1;
    }
/*line 8*/ if(count > ______)
/*line 9*/ printf("Majority element: %d\n",____);
    else
      printf("There is no majority element.\n");
}
```

C2. (a) Two *singly linked lists*, L_1 of n_1 nodes and L_2 of n_2 nodes, $n_1, n_2 > 0$, may have common nodes. The addresses of the first nodes of both L_1 and L_2 are known. Design an $O(n_1 + n_2)$ time algorithm to detect the first common node, if it exists. Your algorithm should report the first common node, if it exists, or report that there is no such node.

(b) You are given an array A of size n. You are told that A comprises three consecutive runs – first a run of 'a's, then a run of 'b's and finally a run of 'c's. Moreover, you are provided an index i such that $A[i] = b$. Design an $O(\log n)$ time algorithm to determine the number of 'b's (i.e., length of the second run) in A.

C3. (a) For a positive integer n, let $G = (V, E)$ be a graph, where $V = \{0, 1\}^n$, i.e., V, the set of vertices, has one-to-one correspondence with the set of all n-bit binary strings, and
$E = \{(u, v) \mid u, v \in V,\ u \text{ and } v \text{ differ in exactly one bit position}\}$.

(i) Determine the size of E.

(ii) Show that G is connected.

(b) For a graph $G = (V, E)$ with a source vertex $s \in V$, the level of a vertex $v \in V$, is the least number of edges in a path from s to v. Design an efficient algorithm to compute the level of each vertex in G. What is the time complexity of your algorithm?

C4. (a) Design a context-free grammar for the language consisting of all strings over $\{a, b\}$ that are not of the form ww, for any string w.

(b) Draw a 4-state DFA for the language $L \subseteq \{a, b\}^*$:
$L = \{x :$ the number of times ab appears in x, is even$\}$.

(c) For the alphabet $\Sigma = \{a, b\}$, the enumeration of the strings of $\{a, b\}^*$ in the lexicographic order is the following

$$\{\epsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}.$$

List the first five strings, in lexicographic order, in the complement of $\{a, ab\}^*$.

C5. (a) Consider the following relations:

$$R_1 = (\underline{A}, B, C), \quad R_2 = (\underline{C}, D, F) \text{ and } R_3 = (\underline{E}, B, C).$$

It is desired to write a query to obtain all information in the relations corresponding to $C =$ "Doctor" and $R_1.B = R_3.B$.

(i) Provide an SQL expression corresponding to the above query.

(ii) Discuss with justification the procedure for executing the query which is expected to minimize the size of the intermediate relations.

(b) A car service agency allows car owners to register their vehicles with the agency. The agency provides hiring services of these cars to registered customers. It is desired to create a database to store information about car bookings. Describe a simple Entity-Relationship (ER) model for this application clearly indicating the attributes of the entities and the arity of the different relations. Derive appropriate relational tables from your model.

C6. (a) Assume that initially 1 Megabyte of memory is available to a multi-programming operating system. The operating system itself occupies 250 Kilobytes of memory, and every process that is executed also requires 250 Kilobytes of memory. Assume that the processes are independent.

(i) How much additional memory is required to get more than 99% CPU utilization if each process spends 50% of its time waiting for the I/O operations?

(ii) How much additional memory is required to get more than 99% CPU utilization if each process spends 20% of its time waiting for the I/O operations?

(b) Suppose two processes enter the ready queue with the following properties:

Process P1 has a total of 8 units of work to perform, but after every 2 units of work, it must perform 1 unit of I/O (i.e., the minimum completion time of this process is 12 units). Assume that there is no work to be done following the last I/O operation.

Process P2 has a total of 20 units of work to perform. This process arrives just behind P1. Show the resulting schedule for the shortest-job-first (preemptive) and the round-robin (RR) algorithms. Assume a time slice of 4 units of RR.

Compute the completion time and turnaround time of each process under each of the two algorithms.

C7. (a) Represent -1.0125 in IEEE 754 standard floating point format.

(b) A CPU uses two levels of caches L1 and L2. It executes two types of jobs J1 and J2. Their details are as follows:

- J1 comes with a probability of 0.3 and requires 2000 memory references, all for reading. For J1, there are 100 misses in L1 and 60 misses in L2.
- J2 comes with a probability of 0.7 and requires 3000 memory references, all for reading. For J2, there are 50 misses in L1 and 70 misses in L2

The L1 hit time is 2 cycles and the L2 hit time is 10 cycles. The L2 miss penalty is 200 cycles. What is the average memory access time?

(c) (i) Show the Karnaugh-map of an irreducible four variable Boolean function $f(x_1, x_2, x_3, x_4)$, whose sum-of-products representation consists of the maximum number of minterms, possible.

(ii) Hence, prove that no Boolean function with n variables, when expressed in sum-of-products form, requires more than 2^{n-1} product terms.

C8. (a) A 1 Km long 10 Mbps CSMA/CD LAN has a propagation speed of 200 m/μsec. Data frames are 256 bits long, including 32 bits of header, checksum and other overhead. The first bit slot after a successful transmission is reserved for the receiver to capture the channel in order to send a 32 bit acknowledgement frame. What is the maximum effective data rate, excluding the overhead achievable in the system?

(b) Can the generator polynomial $(x^6 + 1)$ in CRC detect all burst errors of length 7? Justify your answer.

(c) The data bits $1100, 1011, 0111, 0101$ are organized in rows and columns of a 4×4 matrix. The even parity bits of the rows, the columns and the entire matrix are calculated and appended to the matrix in such a way that the parity bits together with the data bits form a 5×5 matrix, as shown below.

					Row parity bits
	1	1	0	0	0
	1	0	1	1	1
	0	1	1	1	1
	0	1	0	1	0
Column parity bits	0	1	0	1	0 $\leftarrow$ Parity bit of matrix

This 5×5 matrix is then transmitted to the receiver, row by row. Assume that only the data bits may be in error, not the parity bits. The receiver will recompute the parity bits, and based on these values determine errors, if any. Describe the type of errors that cannot be detected with this approach.

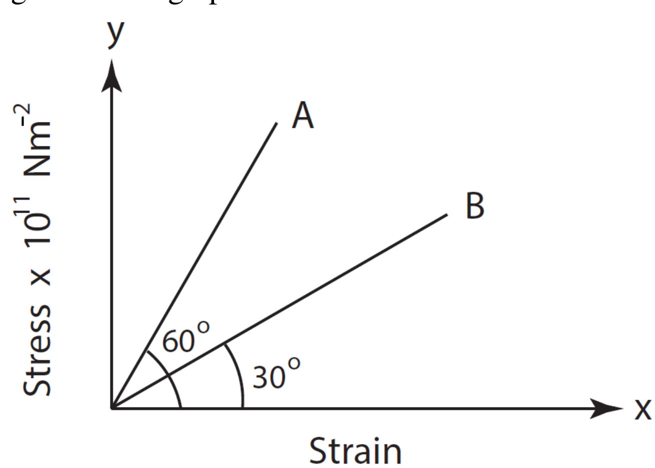
Section II : Engineering and Technology

E1. (a) A mouse weighing $500\ gm$, is sitting at the periphery of a circular turntable, which is stationary but free to rotate about its center. The radius and the moment of inertia of the turntable are $1\ m$ and $20\ kg \cdot m^2$ respectively. Find the angular velocity of the table when the mouse crawls along the periphery at a speed of $10\ cm/sec$.

(b) A man weighing $50\ kg$ is standing on a horizontal conveyor belt moving at a constant speed. If the belt starts accelerating, what is its maximum acceleration for which the man continues to be stationary relative to the belt? Assume $g = 10\ m/s^2$ and the coefficient of friction between the man's shoes and the conveyor belt is 0.1.

(c) A marble is rolling at a constant speed of $1\ m/s$ without slipping. It continues to roll up a $30°$ ramp. How far can the marble roll up along the ramp? Assume $g = 10\ m/s^2$ and the moment of inertia of the marble is $\frac{2}{5}mr^2$, where m and r are its mass and radius, respectively.

E2. (a) A thin rod of length $1\ m$ is hanging horizontally. The rod is supported at both ends by two wires. One wire is made of material A while the other is made of material B. The stress-strain relationship of the materials A and B are given in the graph below:

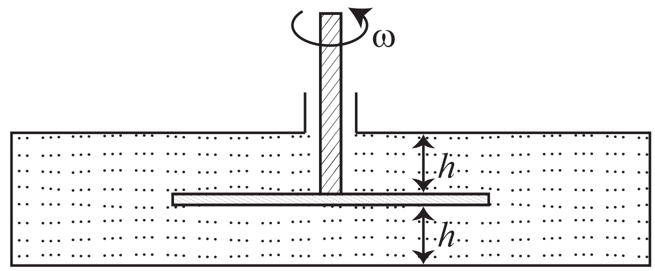


The cross-sectional area of the wire made of A is $5\ mm^2$ whereas that of the wire made of B is $10\ mm^2$.

(i) Find the Young's modulus of material A if the Young's modulus of material B is $10^{11} Nm^{-2}$.

(ii) Find the location of a mass of $100\ gm$ which is to be suspended from the thin rod such that both the wires are subjected to equal stress.

(b) A thin horizontal disc of radius $R = 10\ cm$ is located within a cylindrical cavity filled with oil, as shown in the figure below. The viscosity of the oil is $\eta = 0.08$Poiseuille. The clearance h between the disc and the horizontal planes of the cavity is $1.0\ mm$. Find the power developed by the viscous forces acting on the disc when it rotates with an angular velocity $\omega = 50\ rad/s$. The end effects are to be neglected.



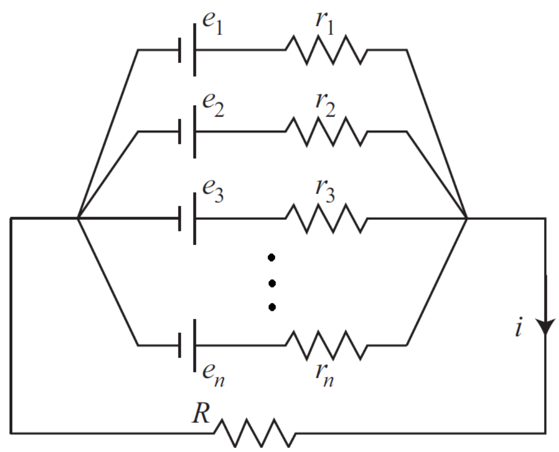
E3. (a) The temperature inside a chamber is to be maintained at $0°C$ for two hours while the outside temperature is $25°C$. The total surface area of the chamber is $4\ m^2$ and the thickness of its wall is 100 cm. The thermal conductivity of the material of the wall is $0.2\ joule/m\ °C$. If ice is placed inside the chamber to keep the chamber at the desired temperature, what is the amount of ice melts in two hours? The latent heat of fusion of ice is $300 \times 10^3\ joule/kg$.

(b) Two Carnot engines C_1 and C_2 are connected in series such that C_1 rejects heat to the sink at temperature T, while C_2 receives heat from the source at the same temperature T. C_1 receives heat from the source at temperature T_1, while C_2 rejects heat to the sink at temperature T_2.

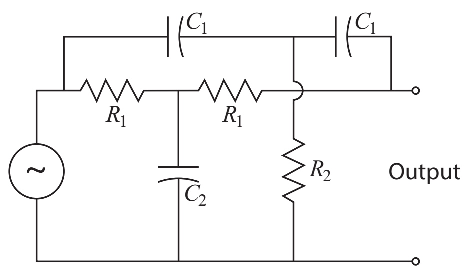
(i) What is the value of T for which the efficiency of C_1 matches with that of C_2?

(ii) Now assume that the efficiency of C_2 is $1/2$. The sink temperature is reduced by $100°C$. The efficiency of C_2 becomes $2/3$. What would be the new temperatures of the source and the sink of C_2?

E4. (a) In the figure below, n non-identical cells of emfs $e, 2e, \cdots, ne$ and the respective internal resistances $r, r/2, \cdots, r/n$ are connected in parallel to deliver a current i to an external resistance R. Derive an expression for the current i. What is the value of i when $n >> 1$ and $R = 2n + 1$?



(b) For the R-C network shown below, find the conditions under which a particular frequency f is rejected at the output. Find an expression for f.



E5. Two d.c. generators A and B supply simultaneously to a common load with same polarity and the resulting voltage across the load is $200\ V$. The generator A has a fixed e.m.f. of $205\ V$ with equivalent internal resistance $0.4\ \Omega$, while the generator B has a fixed e.m.f. of $214\ V$ with equivalent internal resistance $0.5\ \Omega$. Calculate

(a) the percentage of power supplied by the generators A and B to the load,

(b) the total power supplied by the generator B to the system, and

(c) the total copper loss in the system.

E6. A $2200\ kVA$, $440\ V$, $50\ Hz$ transformer has power factor 0.8. The transformer works with maximum efficiency of 88% at half-load. Suppose the transformer is on

- full-load for 8 hours,
- half-load for 6 hours, and
- one-tenth of full-load for the rest of the day.

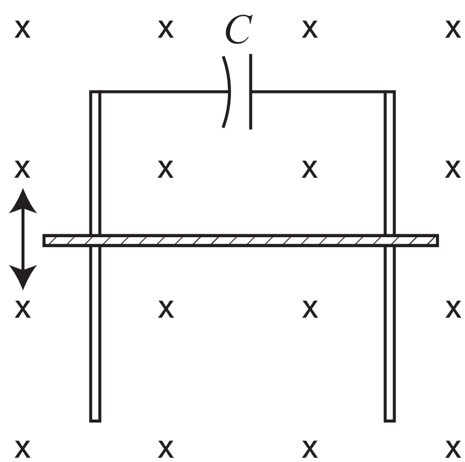
It is measured that the iron loss is $150\ W$ when the transformer works at $220\ V$ and $25\ Hz$. Calculate

(i) the iron loss at the normal voltage and frequency, *i.e.*, $440\ V$ and $50\ Hz$,

(ii) all-day efficiency of the transformer, and

(iii) hysteresis loss and eddy current loss at normal voltage and frequency.

E7. (a) A rod of mass $500\ gm$ and length $20\ cm$ can slide freely on a pair of smooth vertical rails as shown in the figure. A magnetic field $B = 10\ T$ exists in the region perpendicular to the plane of the rails. The rails are connected at the top end by a capacitor $C = 0.5\ F$. Find the acceleration of the rod neglecting any electrical resistance. [Assume $g = 10\ m/s^2$].



(b) Calculate the minimum value of the magnetic field in part (a), for which the rod will stop accelerating downwards.

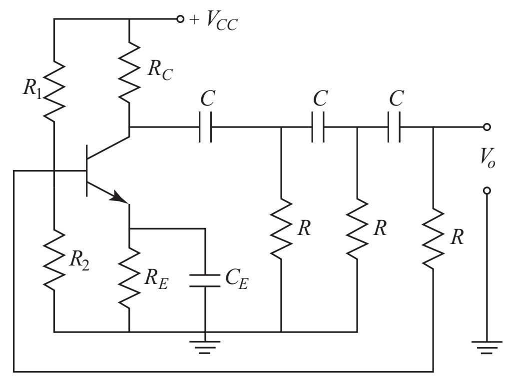
E8. (a) Three identical metal plates P_1, P_2 and P_3 are placed parallel to each other with a distance d_1 between P_1 and P_2, and a distance d_2 between P_2 and P_3.

(i) A charge q is placed on the central plate P_2 while the outer plates P_1 and P_3 are connected by a wire. Calculate the charge on each surface of each of the three plates.

(ii) Repeat your calculations if the charge q is placed on one of the outer plates.

(b) Prove that when the conductivity of an extrinsic silicon sample is minimum, the sample must be slightly p-type. Calculate the electron and hole concentrations when the conductivity is minimum, given that electron mobility $\mu_n = 1350\ cm^2/(V.s)$, hole mobility $\mu_p = 450\ cm^2/(V.s)$ and the intrinsic carrier concentration $n_i = 1.5 \times 10^{10}/cm^3$.

E9. Consider the BJT RC phase shift oscillator shown below:



(a) Prove that the frequency of oscillation is given by

$$\omega_0 = \frac{1}{RC\sqrt{6+4k}},$$

where $k = \frac{R_c}{R}$. Assume that $R >> h_{ie}$, and that h_{re} and h_{oe} are negligible.

(b) Also show that for the BJT, the following condition must be satisfied for stable oscillation:

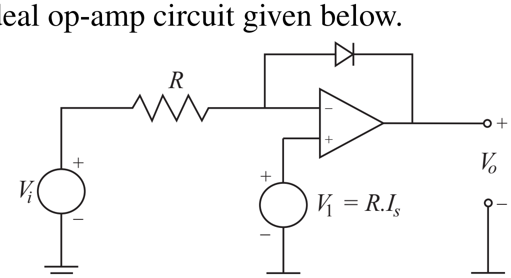
$$h_{fe} \geq 4k + 23 + 29/k.$$

E10. (a) A sequential circuit has two input lines x_1 and x_2 and one output line y. The value of y is 1 if the sequence 011 is fed on the x_1 line while x_2 remains 1 for all 3 cycles. Once y is 1, it remains so until x_2 becomes 0. Construct a minimum-row state table for this circuit.

(b) (i) Draw the Karnaugh-map of an irreducible four-variable Boolean function $f(x_1, x_2, x_3, x_4)$ whose sum-of-products representation consists of the maximum possible number of minterms.

(ii) Hence prove that no Boolean function with n variables, when expressed in sum-of-products form, requires more than 2^{n-1} product terms.

E11. (a) Consider the ideal op-amp circuit given below.



The current through the diode at temperature T is given by:
$I = I_s(e^{\frac{V}{V_T}} - 1)$ where V is the forward voltage across the diode and V_T is the volt equivalent of temperature.

(i) Obtain an expression for V_o as a function of V_i, for $V_i > 0$.

(ii) If $R = 100\ k\Omega$, $I_s = 1\ \mu A$ and $V_T = 25\ mV$, find the input voltage V_i for which $V_o = 0$. [You may use $e^4 = 54.6$]

(b) Draw an R-$2R$ ladder D/A converter for 4 input bits. Derive an expression for its output voltage in terms of R and V_R, where V_R is the voltage across a resistor R when the corresponding input bit is 1.

E12. Given a set of n unsorted real numbers in an array A, the following C program computes a pair of elements in the array A, say $A[index1]$ and $A[index2]$, such that $|A[index1] - A[index2]|$ is the minimum $|A[i] - A[j]|$ among all pairs $(i, j), i, j = 1, 2, \ldots, n; i \neq j$. Such a pair ($A[index1]$, $A[index2]$) is called the closest pair.

```
main()
{
int A[10000],n,i,index1, index2;
void compare(int*,int,int,int);

printf("\n Enter the number of elements :");
scanf("%d",&n);
printf("\n Enter the numbers : ");
for (i=0;i<n;i++)
  scanf("%d",&A[i]);

compare(A,n,index1,index2);

printf("The closest pair is ");
printf("%d %d\n",A[index1],A[index2]);
printf("\n");
}
```

(a) Write an efficient C code for the function `compare(A,n,i,j)`.

(b) For an array of n distinct elements, what is the maximum number (in terms of n) of comparisons that your function `compare` will make?

[Note: Your marks will depend not only on the correctness of the code but also on the run-time efficiency of your code].

Section III : Mathematics

M1. Let $f : [a, b] \to [a, b]$ be a real-valued function such that

$$|f(x) - f(y)| < |x - y|, \text{ for all } x, y \in [a, b], x \neq y.$$

Define $g(x) = |f(x) - x|$ over $[a, b]$.

(a) Show that g is continuous.

(b) Hence, or otherwise, show that f has a unique fixed point.

M2. (a) Suppose $x_1(t)$ and $x_2(t)$ are two linearly independent solutions of the equations:

$$x_1'(t) = 3x_1(t) + 2x_2(t) \text{ and } x_2'(t) = x_1(t) + 2x_2(t),$$

where $x_1'(t), x_2'(t)$ denote the first derivative of functions $x_1(t)$ and $x_2(t)$ respectively with respect to t. Find the general solution of

$$x''(t) - 5x'(t) + 4x(t) = 0$$

in terms of $x_1(t)$ and $x_2(t)$.

(b) Using the transformation $x(t) = ty(t)$, show that the equation

$$tx''(t) - 2x'(t) + \frac{4}{t}x(t) = 0, \quad t \geq 1$$

can be reduced to $t^2y''(t) + 2y(t) = 0$. Hence, find the general solution.

M3. (a) We define a real valued function f as

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is continuous.

(b) Let $g(x)$ be a continuous function such that

$$g'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad \text{for all } x \neq 0.$$

Justify whether g is differentiable at 0 or not.

M4. (a) Let $x_n = \sqrt{x_{n-1}y_{n-1}}$ and $y_n = \frac{x_n + y_{n-1}}{2}$, $n \geq 2$ and $x_1 = 1, y_1 = 2$. Prove that $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist and both are equal.

(b) Let $(0, 1)$ denote the open interval from 0 to 1. Let f be an injective function on $(0, 1)$ to itself. Show that f has neither a minimum nor a maximum.

M5. (a) Show that any finite acyclic digraph has a vertex of indegree zero.

(b) Let G be a connected graph of n vertices such that both G and $\overline{G}$ are connected. Further, both have a clique and an independent set of size 4.

(i) Give an example of G with $n = 7$.

(ii) Show that for any such graph, $n \geq 7$.

M6. (a) Let R be an integral domain with unity. Let $\alpha, w \in R$ be such that $\alpha^2 = w$, $w^n = 1$ and $w^i \neq 1$ for any $0 < i < n$, for some even integer n. Show that $\alpha^n = -1$.

(b) Let $\mathbb{F}_2$ be the field with two elements and $\tau(x) = x^4 + x + 1$.

(i) Show that $\mathbb{F} = \mathbb{F}_2[x]/\langle \tau(x) \rangle$ is a field.

(ii) Define $A = ((a_{i,j}))_{0 \leq i,j \leq 2}$ to be a 3×3 matrix with entries from $\mathbb{F}$ where $a_{i,j} = x^i(1+x)^j$. Determine whether A is invertible.

M7. (a) Let G be a group such that for all $x, y, z \in G$, $xy = yz \Rightarrow x = z$. Show that G is abelian.

(b) Prove that a group G has exactly three subgroups if and only if $|G| = p^2$ for a prime p.

(c) Let S be a finite ring with no zero divisors. Define

$$Z = \{x \in S : xr = rx, \forall r \in S\}$$

and suppose $|Z| = q$. Show that $|S| = q^n$ for some $n \geq 1$.

M8. (a) Let A be a 4×4 matrix such that the sum of the entries in each row of A equals 1. Find the sum of all entries in the matrix A^5.

(b) Let

$$J_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n \times n}$$

be the $n \times n$ matrix whose all entries are 1 and I_n denotes the identity matrix of size n.

(i) Show that, whenever the matrix $I_n - \lambda J_n$ is invertible, the inverse is also of the form $I_n - \lambda' J_n$ for some λ' where $\lambda, \lambda' \in \mathbb{R}$.

(ii) Find all values of λ for which the matrix $I_n - \lambda J_n$ is invertible.

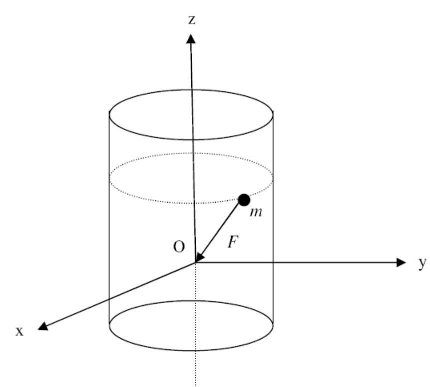
Section IV : Physics

P1. (a) A mass $2m$ is suspended from a fixed support by a spring of spring constant $2k$. From this mass another mass m is suspended by another spring of spring constant k.

(i) Derive the Lagrangian of the coupled system.

(ii) Hence, find the equation of motion of the coupled system.

(b) A particle of mass m is constrained to move on the surface of a cylinder, as shown in the figure below. The particle is subjected to a force F directed towards the origin O, and proportional to the distance of the particle from the origin. Construct the Hamiltonian and Hamilton's equation of motion.



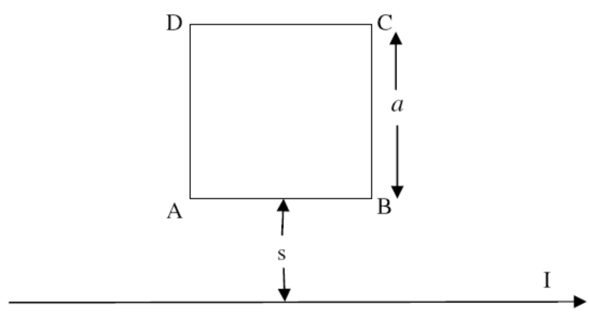
(c) A train with proper length L moves at a speed $c/2$ with respect to the ground, where c is the speed of light. A person standing at the back-end of the train throws a ball towards the front-end of the train. The ball moves at a speed $c/3$ with respect to the train. How much time does the ball take to reach the front-end of the train? What distance does the ball cover with respect to the ground frame?

P2. (a) Which one of the following electrostatic fields is infeasible? Justify your answer. Assume k is a constant with appropriate units.

$$E_1 = k[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}], \quad E_2 = k[y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}]$$

For the feasible one, find the potential at any point (x_0, y_0, z_0) using the origin $(0, 0, 0)$ as your reference point.

(b) A square loop $ABCD$ of wire (having side of length a) lies on a table. A long straight wire, carrying a current I, is on the same plane of the loop and parallel to AB at a distance s from AB (see the figure below).



(i) Find the flux through the loop.

(ii) If the loop is pulled away from the wire at a speed v, what e.m.f. is generated?

P3. (a) A particle of mass m and energy E follows the potential energy function $V(x)$ in the x-direction, where

$$V(x) = \begin{cases} 0, & x \leq 0 \\ V_0, & x > 0 \end{cases}$$

Calculate

(i) the reflection coefficient for $E < V_0$ and

(ii) the transmission coefficient for $E > V_0$.

(b) The normalized wavefunction of a particle is $\psi(x,t) = Ae^{i(ax-bt)}$, where A, a and b are constants. Evaluate the uncertainty in its momentum.

P4. (a) A particle of mass m strikes a stationary nucleus of mass M and activates an endoergic reaction. Show that the threshold kinetic energy required to initiate this reaction is :

$$E_{th} = \frac{m + M}{M} \mid Q \mid,$$

where Q is the energy of the reaction.

(b) A radionuclide with half-life of 5 days is produced in a reactor at the rate of 2×10^9 nucleii per second. How soon after the beginning of production of that radionuclide will its activity be 10^9 becquerels ?

P5. (a) The primitive translation vectors of the hexagonal space lattice of side a may be taken as:

$$\vec{a}_1 = \frac{\sqrt{3}a}{2}\hat{x} + \frac{a}{2}\hat{y}; \quad \vec{a}_2 = -\frac{\sqrt{3}a}{2}\hat{x} + \frac{a}{2}\hat{y}; \quad \vec{a}_3 = c\hat{z}.$$

(i) Find the volume of the primitive cell.

(ii) Show that the primitive translations of the reciprocal lattice are

$$\vec{b}_1 = \frac{2\pi}{\sqrt{3}a}\hat{x} + \frac{2\pi}{a}\hat{y}; \quad \vec{b}_2 = -\frac{2\pi}{\sqrt{3}a}\hat{x} + \frac{2\pi}{a}\hat{y}; \quad \vec{b}_3 = \frac{2\pi}{c}\hat{z}.$$

(iii) Draw the first Brillouin Zone of the hexagonal space lattice.

(b) Consider an open-circuited step-graded p-n junction. Suppose the p-type side has a uniform concentration N_A of acceptor impurity atoms and the n-type side has a uniform density N_D of donor impurity atoms. If the junction is kept at temperature T and the intrinsic carrier concentration is n_i, show that the built-in potential between the p-type and n-type sides can be approximated as

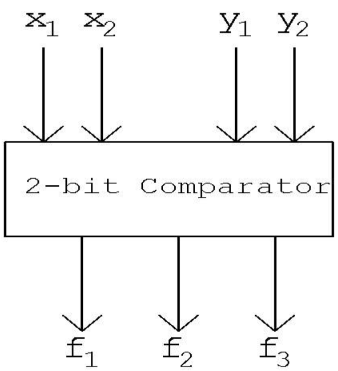
$$V_0 \approx \frac{K_B T}{q} ln(\frac{N_A N_D}{n_i^2})$$

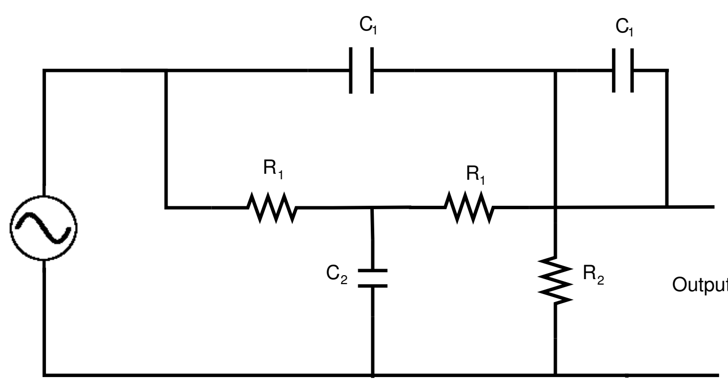
where K_B is Boltzman's constant and q is one electronic charge.

P6. (a) An n-bit comparator is a circuit which compares the magnitude of two n-bit numbers X and Y. Figure below shows a 2-bit comparator, with three outputs such that

- $f_1 = 1$ if and only if $X > Y$,
- $f_2 = 1$ if and only if $X = Y$, and
- $f_3 = 1$ if and only if $X < Y$.

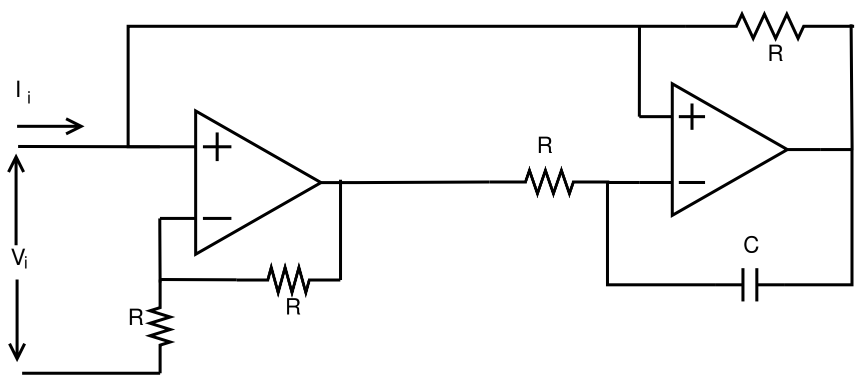
Design the logic circuits only using AND, OR and NOT gate to produce the three outputs f_1, f_2 and f_3.



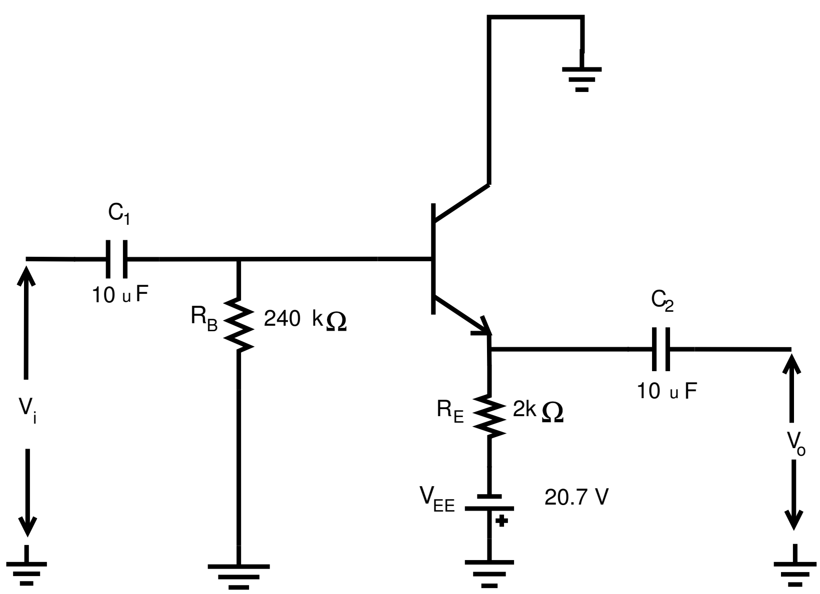


(b) Given the RC network shown below, find the conditions under which a particular frequency f is rejected at the output. Find the expression for f.

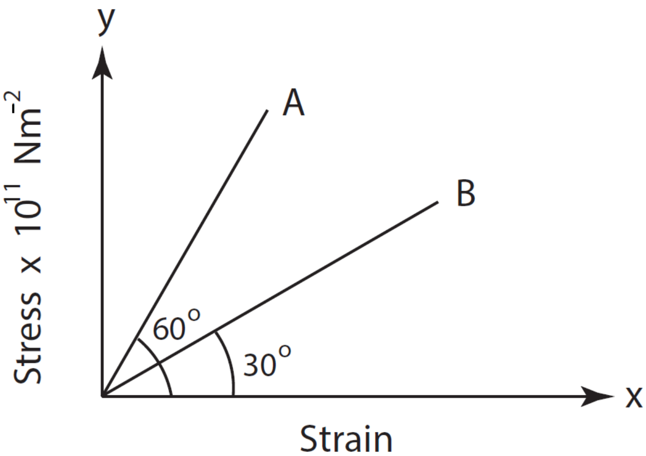
P7 (a) Prove that the input impedance of the following op-amp circuit is inductive.



(b) For the BJT circuit shown below, determine the collector to emitter voltage V_{CE_Q} at operating point and also find the emitter current I_E. Assume $\beta = 79$.



P8. (a) A thin rod of length $1\ m$ is hanging horizontally. The rod is supported at both ends by two wires. One wire is made of material A while the other is made of material B. The stress-strain relationship of the materials A and B are given in the graph below:



The cross-sectional area of wire A is $5\ mm^2$ whereas that of wire B is $10\ mm^2$.

(i) Find the Young's modulus of material A if the Young's modulus of material B is $10^{11} Nm^{-2}$.

(ii) Find the location of a mass of $100\ gm$ which is to be suspended from the thin rod such that both the wires are subjected to equal stress.

(b) Two Carnot engines C_1 and C_2 are connected in series such that C_1 rejects heat to the sink at temperature T, while C_2 receives heat from the source at the same temperature T. C_1 receives heat from the source at temperature T_1, while C_2 rejects heat to the sink at T_2.

(i) What is the value of T for which the efficiency of C_1 matches with that of C_2?

(ii) Now assume that the efficiency of C_2 is $1/2$. The sink temperature is reduced by $100°C$. The efficiency of C_2 becomes $2/3$. What would be the new temperatures of the source and the sink of C_2?

Section III : STATISTICS

S1. Two numbers l and b are chosen randomly and independently from $[0, 10]$. A rectangle is constructed with l and b as its length and breadth respectively. Find the probability that the length of its diagonal is less than or equal to 10.

S2. Let U and V be possibly dependent discrete random variables uniformly distributed on $\{1, 2, \ldots, K\}$. Let W be another discrete random variable distributed uniformly on $\{1, 2, \ldots, K\}$, independently of U and V. Let $X = (V + W) \bmod K$. Show that

(a) X too is uniformly distributed on $\{1, 2, \ldots, K\}$;

(b) U and X are independent.

S3. Consider the linear model

$$Y_i = \beta_0 + \beta_1 \left(\frac{i}{n}\right) + \epsilon_i,\ i = 1, 2, \ldots n\ (n \geq 3),$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independently distributed $N(0, \sigma^2)$ variables. $\beta_0, \beta_1, \sigma^2$ are unknown parameters such that $\beta_0,\ \beta_1 \in (-\infty, \infty)$ and $\sigma^2 \in (0, \infty)$.

(a) Find the least squares estimators of β_0 and β_1.

(b) Are the least squares estimators of β_0 and β_1 (based on the observed values of Y_i's) UMVUE? Justify your answer.

S4. Consider a random variable X that has a uniform distribution over $(0, 2\beta)$, where $\beta > 0$. Let $Y = \max(X, 2\beta - X)$.

(a) Find μ_Y, the expectation of Y.

(b) Let $X_1,\ X_2, \ldots, X_n$ be a random sample from the above distribution, β being unknown. Find two distinct unbiased estimators of μ_Y obtained in (a), based on the entire sample.

S5. A population contains 100 units labeled $u_1, u_2, \ldots, u_{100}$. Let Y_i denote the value of Y, the variable under study, corresponding to the population unit u_i, $i = 1, 2, \ldots, 100$. For estimating the population mean $\bar{Y}$, a sample of size 10 is drawn in the following manner:

i. a simple random sample of size 8 is drawn without replacement from the 98 units $u_2, u_3, \ldots, u_{99}$;

ii. the sample drawn in step (i) is augmented by the units u_1 and u_{100}.

Based on the sample of size 10 so obtained, suggest an unbiased estimator of $\bar{Y}$ and obtain its variance.

S6. Let X be a random variable having a $N(\mu, 1)$ distribution. Define $Y = e^X$.

(a) Derive the probability density function of Y.

(b) Find μ_Y, the expectation of Y. Let $Y_1,\ Y_2, \ldots, Y_n$ be a random sample from the distribution of Y. Find the maximum likelihood estimator of μ_Y based on this sample.

S7. A straight line regression $E(y) = \alpha + \beta x$ is to be fitted using four observations. Assume $Var(y|x) = \sigma^2$ for all x. The values of x should be chosen from the interval $[-1, 1]$. The following choices of the values of x are available:

(a) two observations at each of $x = -1$ and $x = 1$,

(b) one observation at each of $x = -1$ and $x = 1$, and two observations at $x = 0$,

(c) one observation at each of $x = -1, -1/2, 1/2, 1$.

If the goal is to estimate the slope with least variance, which of the above choices would you recommend and why?

S8. Let $X_1,\ X_2,\ldots,X_n$ be independent random variables identically distributed as

$$f(x)=\frac{1}{\theta}e^{x/\theta},\ \theta>0.$$

For the problem of testing $H_0:\theta=1$ against $H_1:\theta=2$, consider two possible tests with the respective critical regions

$$\begin{aligned}\omega_1 &= \left\{\sum_{i=1}^{n} X_i \geq c_1\right\}\\ \omega_2 &= \{T\geq c_2\}\,.\end{aligned}$$

where T is the number of X_i's having values greater than or equal to 2.

(a) If n is large, determine the approximate values of c_1 and c_2 for which the respective tests are of size α.

(b) Which of the two tests will require more observations to achieve the same power? Justify your answer.

—x—