Solving Recurrence Relations

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We want to solve the recurrence relation

$$a_n = Aa_{n-1} + Ba_{n-2}$$

where A and B are real numbers. The solutions depend on the nature of the roots of the characterstic equation

$$s^2 - As - B = 0 \tag{1}$$

We consider three cases for the roots of (1).

1. If we have two distinct real roots s_1 and s_2, then

$$a_n = \alpha s_1^n + \beta s_2^n.$$

2. If we have exactly one real root s, then

$$a_n = \alpha s^n + \beta n s^n.$$

3. If we have two complex conjugate roots in polar form $s_1 = r\angle\theta$ and $s_2 = r\angle(-\theta)$, then

$$a_n = r^n\big(\alpha\cos(n\theta) + \beta\sin(n\theta)\big).$$

In all cases, the numbers α and β can be determined if we are given the values of a_0 and a_1.

Example 1. Consider the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} \tag{2}$$

with initial conditions $a_0 = 1$ and $a_1 = 4$. The characteristic equation is

$$s^2 - 5s + 6 = (s-2)(s-3) = 0.$$

Since the roots are $s = 2$ and $s = 3$, any solution of (2) has the form $a_n = \alpha\, 3^n + \beta\, 2^n$. Therefore,

$$\begin{aligned} a_0 &= \alpha + \beta = 1 \\ a_1 &= 3\alpha + 2\beta = 4. \end{aligned}$$

Solving this linear system, we get $\alpha = 2$ and $\beta = -1$. The solution of (2) with the given initial conditions is then

$$\boxed{a_n = 2\cdot 3^n - 2^n}$$

Example 2. Consider the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2} \tag{3}$$

with initial conditions $a_0 = 4$ and $a_1 = 6$. The characteristic equation is

$$s^2 - 6s + 9 = (s-3)^2 = 0.$$

Since $s = 3$ is the only root, any solution of (3) has the form $a_n = \alpha\, 3^n + \beta\, n3^n$. Therefore,

$$\begin{aligned} a_0 &= \alpha = 4 \\ a_1 &= 3\alpha + 3\beta = 6. \end{aligned}$$

Solving this system, we get $\alpha = 4$ and $\beta = -2$. The solution of (3) with its initial conditions is then

$$\boxed{a_n = 4 \cdot 3^n - 2n3^n}$$

Example 3. Consider the recurrence relation

$$a_n = 2a_{n-1} - 2a_{n-2} \tag{4}$$

with initial conditions $a_0 = 1$ and $a_1 = 3$. The characteristic equation is

$$s^2 - 2s + 2 = (s-1)^2 + 1 = 0.$$

We have two complex conjugate roots $s_1 = 1 + i$ and $s_1 = 1 - i$. In polar form $s_1 = r\angle\theta$ with $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$. Any solution of (4) has the form $a_n = (\sqrt{2})^n \big(\alpha\, \cos(n\frac{\pi}{4}) + \beta\, \sin(n\frac{\pi}{4})\big)$. Therefore,

$$\begin{aligned} a_0 &= \alpha = 1 \\ a_1 &= \sqrt{2}\left(\alpha\frac{1}{\sqrt{2}} + \beta\frac{1}{\sqrt{2}}\right) = 3. \end{aligned}$$

Solving this system, we get $\alpha = 1$ and $\beta = 2$. The solution of (4) with its initial conditions is then

$$\boxed{a_n = (\sqrt{2})^n \big(\cos(n\tfrac{\pi}{4}) + 2\,\sin(n\tfrac{\pi}{4})\big)}$$

Example 4. Consider the sequence of Fibonacci numbers that satisfy the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \tag{5}$$

with initial conditions $f_0 = 0$ and $f_1 = 1$. The characteristic equation is

$$s^2 - s - 1 = 0.$$

The roots are $s = (1+\sqrt{5})/2$ and $s = (1-\sqrt{5})/2$. Then, any solution of (5) has the form

$$f_n = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Therefore,

$$\begin{aligned} f_0 &= \alpha + \beta = 0 \\ f_1 &= \alpha\left(\frac{1+\sqrt{5}}{2}\right) + \beta\left(\frac{1-\sqrt{5}}{2}\right) = 1. \end{aligned}$$

Solving this linear system, we get $\alpha = 1/\sqrt{5}$ and $\beta = -1/\sqrt{5}$. Therefore, the Fibonacci numbers are obtained by the following formula, commonly known as Binet's formula.

$$\boxed{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

A recurrence relation of the form

$$a_n = Aa_{n-1} + Ba_{n-2} + F(n)$$

for $F(n)$ not identically zero is said to be **nonhomogeneous**. Its associated **homogeneous** recurrence relation is

$$a_n = Aa_{n-1} + Ba_{n-2}.$$

Theorem 1. *If $a_n^{(p)}$ is a particular solution of the nonhomogeneous recurrence relation*

$$a_n = Aa_{n-1} + Ba_{n-2} + F(n), \tag{6}$$

then every solution of (6) is of the form

$$a_n = a_n^{(h)} + a_n^{(p)},$$

where $a_n^{(h)}$ is a solution of the associated homogeneous recurrence relation.

To apply this theorem, we need to find a particular solution of (6). This is a difficult problem in general but a standard technique exists for simple types of $F(n)$ such as

- Polynomial, e.g. $F(n) = 5n^2 - 2n + 1$.
- Exponential, e.g. $F(n) = 3^n$.
- Exponential $\times$ Polynomial, e.g. $F(n) = 2^n(5n^2 + 3n + 1)$.

Theorem 2. *Consider the nonhomogeneous recurrence relation*

$$a_n = Aa_{n-1} + Ba_{n-2} + F(n),$$

where $F(n) = t^n$(Polynomial of degree N). If t is not a root of $s^2 - As - B = 0$, then there is a particular solution of the form

$$a_n^{(p)} = t^n(p_0 + p_1 n + p_2 n^2 + \cdots + p_N n^N).$$

If t is a root of $s^2 - As - B = 0$ of multiplicity m, then there is a particular solution of the form

$$a_n^{(p)} = t^n n^m(p_0 + p_1 n + p_2 n^2 + \cdots + p_N n^N).$$

Note that if $F(n)$ is simply a polynomial like $F(n) = 5n^2 - 2n + 1$, then $t = 1$ in the above theorem.

Example 4. Consider the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + F(n).$$

The characteristic equation of its associated homogeneous equation is

$$s^2 - 5s + 6 = (s-2)(s-3) = 0.$$

1. If $F(n) = 2n^2$, then a particular solution has the form $a_n^{(p)} = An^2 + Bn + C$.
2. If $F(n) = 5^n(3n^2 + 2n + 1)$, then $a_n^{(p)} = 5^n(An^2 + Bn + C)$.
3. If $F(n) = 5^n$, then $a_n^{(p)} = 5^n A$.
4. If $F(n) = 3^n$, then $a_n^{(p)} = 3^n An$.
5. If $F(n) = 2^n(3n + 1)$, then $a_n^{(p)} = 2^n n(An + B)$.

The values of the constants A, B, and C can be found by substituting $a_n^{(p)}$ in the recurrence relation.

Example 5. For the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n),$$

the characteristic equation of its associated homogeneous equation is

$$s^2 - 6s + 9 = (s-3)^2 = 0.$$

1. If $F(n) = 3^n$, then $a_n^{(p)} = 3^n An^2$.
2. If $F(n) = 3^n(5n+1)$, then $a_n^{(p)} = 3^n n^2(An+B)$.
3. If $F(n) = 2^n(5n+1)$, then $a_n^{(p)} = 2^n(An+B)$.

Example 6. For the recurrence relation

$$a_n = 3a_{n-1} - 2a_{n-2} + F(n),$$

the characteristic equation of its associated homogeneous equation is

$$s^2 - 3s + 2 = (s-1)(s-2) = 0.$$

If $F(n) = 3n+1$, then a particular solution has the form $a_n^{(p)} = n(An+B)$.

To see this, observe that $F(n) = 1^n(3n+1)$ and $s = 1$ is a root of multiplicity one of the characteristic equation.

Finally, here are two basic recurrence relations.

Theorem 3. *(Arithmetic sequence) If $a_n = a_{n-1} + d$ and $a_0 = \alpha$, then $a_n = dn + \alpha$.*

Theorem 4. *(Geometric sequence) If $a_n = ka_{n-1}$ and $a_0 = \alpha$, then $a_n = \alpha k^n$.*