Theory of Algorithms. Spring 2000. Homework 6 Solutions.

Section 4.3

(3) Let $L = \{\, w \in \{a,b\}^* \,:\, n_a(w) = n_b(w) \,\}$. Then L is not regular. (Since $L^* = L$, L^* is also not regular either.)

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^m b^m$. Notice that $w \in L$ and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^k$ for some k with $1 \leq k \leq m$. Now let $i = 2$. Then $w_i = w_2 = xy^2z = a^{m+k}b^m$. So $w_i \notin L$ because $m + k \neq m$. This contradicts the Pumping Lemma. So L is not regular. □

(4a) Let $L = \left\{\, a^n b^l a^k \,:\, k \geq n + l \,\right\}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^m b^m a^{2m}$. Notice that $w \in L$ and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^t$ for some t with $1 \leq t \leq m$. Now let $i = 2$. Then $w_i = w_2 = xy^2z = a^{m+t}b^m a^{2m}$. So $w_i \notin L$ because $2m \ngeq (m+t) + m$. This contradicts the Pumping Lemma. So L is not regular. □

(4b) Let $L = \left\{\, a^n b^l a^k \,:\, k \neq n + l \,\right\}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Then $\bar{L} \cap L(a^*b^*a^*)$ is also regular since the family of regular languages is closed under compliment and intersection. Let us write L_1 for $\bar{L} \cap L(a^*b^*a^*)$. Notice that $L_1 = \left\{\, a^n b^l a^k \,:\, k = n + l \,\right\}$. We will apply the Pumping Lemma to L_1. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^m b^m a^{2m}$. Notice that $w \in L_1$ and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^t$ for some t with $1 \leq t \leq m$. Now let $i = 2$. Then $w_i = w_2 = xy^2z = a^{m+t}b^m a^{2m}$. So $w_i \notin L_1$ because $2m \neq (m+t) + m$. This contradicts the Pumping Lemma. So L is not regular. □

(4c) Let $L = \left\{\, a^n b^l a^k \,:\, n = l \text{ or } l \neq k \,\right\}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^m b^m a^m$. Notice that $w \in L$ (since n=m=l) and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^t$ for some t with $1 \leq t \leq m$. Now let $i = 2$. Then $w_i = w_2 = xy^2z = a^{m+t}b^m a^m$. So $w_i \notin L$ because $n = m + t \neq m = l$ and $l = m = k$. This contradicts the Pumping Lemma. So L is not regular. □

(4d) Let $L = \left\{\, a^n b^l \,:\, n \leq l \,\right\}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^m b^m$. Notice that $w \in L$ and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^t$ for some t with $1 \leq t \leq m$. Now let $i = 2$. Then $w_i = w_2 = xy^2z = a^{m+t}b^m$. So $w_i \notin L$ because $m + t \nleq m$. This contradicts the Pumping Lemma. So L is not regular. □

(4e) Let $L = \{\, w \in \{a,b\}^* \,:\, n_a(w) = n_b(w) \,\}$. Then L is not regular.

Proof. If L were regualr then $\bar{L}$ would be regular. But we proved in exercise (3) above that $\bar{L}$ is not regular. ☐

(4f) Let $L = \{\, ww \,:\, w \in \{a,b\}^* \,\}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^m b a^m b$. Notice that $w \in L$ and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^k$ for some k with $1 \leq k \leq m$. Now let $i = 2$. Then $w_i = w_2 = xy^2z = a^{m+k}ba^mb$. So $w_i \notin L$. This contradicts the Pumping Lemma. So L is not regular. ☐

(5a) We did this one in class.

(5b) This follows from 5a since the family of regular languages is closed under compliments.

(5c) Let $L = \{\, a^n \,:\, n = k^2 \text{ for some } k \geq 0 \,\}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^{m^2}$. Notice that $w \in L$ and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^t$ for some t with $1 \leq t \leq m$. Now let $i = 2$. Then $w_i = w_2 = xy^2z = a^{m^2+t}$. Now $m^2 + t \leq m^2 + m < m^2 + 2m + 1 = (m+1)^2$. So $m^2 + t \neq k^2$ for any k. So $w_i \notin L$. This contradicts the Pumping Lemma. So L is not regular. ☐

(5d) Let $L = \left\{\, a^n \,:\, n = 2^k \text{ for some } k \geq 0 \,\right\}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^{2^m}$. Notice that $w \in L$ and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^t$ for some t with $1 \leq t \leq m$. Now let $i = 2$. Then $w_i = w_2 = xy^2z = a^{2^m+t}$. Now $2^m + t \leq 2^m + m < 2^m + 2^m = 2(2^m) = 2^{m+1}$. So $2^m + t \neq 2^k$ for any k. (In the above calculation we use the fact that, since $m \geq 1$, $m < 2^m$. This can be proved by induction on m.) So $w_i \notin L$. This contradicts the Pumping Lemma. So L is not regular. ☐

(8) Consider the statement: "If L_1 and L_2 are nonregular languages, then $L_1 \cup L_2$ is nonregular." This statement is **FALSE**. For example let L_1 be the L from exercise ($5d$) above. So L_1 is nonregular. Let $L_2 = \{a\}^* - L_1$. Since the family of regular languages is closed under compliment, L_2 is also nonregular. But $L_1 \cup L_2 = \{a\}^*$ which, of course, is regular.

(9a) Let $L = \left\{\, a^n b^l a^k \,:\, n + l + k > 5 \,\right\}$. Then L is regular. Here is a regular expression for L:

$$aaaaaaa^*b^*a^* + aaaaabb^*a^* + aaaabbb^*a^* + aaaabaa^* + aaabbbb^*a^* + aaabbaa^* + aaabaaa^* +$$
$$+aabbbbb^*a^* + aabbbaa^* + aabbaaa^* + aabaaaa^* + abbbbbb^*a^* + abbbbaa^* + abbbaaa^* + abbaaaa^* + abaaaaa^* +$$
$$bbbbbbb^*a^* + bbbbbaa^* + bbbbaaa^* + bbbaaaa^* + bbaaaaa^* + baaaaaa^*$$

(9b) Let $L = \{ a^n b^l a^k : n > 5, l > 3, k \le l \}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^6 b^{4m} a^{4m}$. Notice that $w \in L$ and $|w| \ge m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that there are three cases for what y looks like. Either (i) $y = a^t$ for some t with $1 \le t \le 6$; or (ii) $y = b^t$ for some t with $1 \le t \le m$; or (iii) $y = a^t b^s$ for some t and s with $1 \le t \le 6$ and $1 \le s \le m$. In Case (i), let $i = 0$. Then $w_i = w_0 = xz = a^{6-t} b^{4m} a^{4m}$. Then $w_i \notin L$ since $6 - t$ is not greater than 5. In Case (ii) let $i = 0$. Then $w_i = w_0 = xz = a^6 b^{4m-t} a^{4m}$. Then $w_i \notin L$ since it is not the case that $4m \le 4m - t$. In Case (iii) let $i = 2$. Then $w_i = w_2 = xy^2 z = a^6 b^s a^t b^s z$. So again $w_i \notin L$. This contradicts the Pumping Lemma. So L is not regular. □

(9c) Let $L = \{ a^n b^l : n/l \text{ is an integer.} \}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^{m+1} b^{m+1}$. Notice that $w \in L$ and $|w| \ge m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^k$ for some k with $1 \le k \le m$. Now let $i = 2$. Then $w_i = w_2 = xy^2 z = a^{m+k+1} b^{m+1}$. Now $m + k + 1 \le m + m + 1 < 2m + 2 = 2(m + 1)$. So $m + k + 1$ is not a multiple of $m + 1$. So $(m + k + 1)/(m + 1)$ is not an integer. So $w_i \notin L$. This contradicts the Pumping Lemma. So L is not regular. □

(9d) Let $L = \{ a^n b^l : n + l \text{ is a prime number.} \}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Let p be the least prime number greater than m. Then let $w = a^p b^0 = a^p$. Notice that $w \in L$ and $|w| \ge m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^k$ for some k with $1 \le k \le m$. Now let $i = p + 1$. Then $w_i = a^{p+pk}$. Now $p + pk = p(k + 1)$ is not a prime number. So $w_i \notin L$. This contradicts the Pumping Lemma. So L is not regular. □

(9e) Let $L = \{ a^n b^l : n \le l \le 2n \}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = a^m b^m$. Notice that $w \in L$ (since $m \le m \le 2m$) and $|w| \ge m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Notice that $y = a^k$ for some k with $1 \le k \le m$. Now let $i = 2$. Then $w_i = w_2 = xy^2 z = a^{m+k} b^m$. Then $w_i \notin L$ since it is not the case that $m + k \le m$. This contradicts the Pumping Lemma. So L is not regular. □

(9f) Let $L = \{ a^n b^l : n \ge 100, \ l \le 100 \}$. Then L is regular. Here is a regular expression for L:

$$a^{100} a^* (\lambda + b + bb + bbb + bbbb + bbbbb + \cdots + b^{98} + b^{99} + b^{100}).$$

(11) Let L_1 and L_2 be regular languages. Let $L = \{ w : w \in L_1, w^R \in L_2 \}$. Then L is regular. To see this, just notice that $L = L_1 \cap L_2^R$. Since the family of regular languages is closed under reversal and intersection, L is regular.

(13a) Let $L = \{ uww^Rv : u, v, w \in \{a, b\}^+ \}$. Then L is regular. Let r be the following regular expression.

$$(a + b)(a + b)^*(aa + bb)(a + b)(a + b)^*.$$

Claim. $L = L(r)$.

Proof. First we will show that $L \subseteq L(r)$. Let $x \in L$. So then $x = uww^Rv$ for some $u, v, w \in \{a, b\}^+$. Suppose the last symbol of w is a. (If the last symbol of w is b the proof is similar.) Let us write $w = ya$ with $y \in \{a, b\}^*$. Then we can write $x = uyaay^Rv$. Now $uy \in L\big((a + b)(a + b)^*\big)$ and $y^Rv \in L\big((a + b)(a + b)^*\big)$ so $x \in L(r)$.

Next we will show that $L(r) \subseteq L$. Let $x \in L(r)$. So then $x = uaav$ or $x = ubbv$ with $u, v \in \{a, b\}^+$. In either case we can write $x = uww^Rv$ with $u, v, w \in \{a, b\}^+$. So $x \in L$. □

(13b) Let $L = \{ uss^Rv : u, v, s \in \{a, b\}^+ ,\ |u| \geq |v| \}$. Then L is not regular.

Proof. Assume towards a contradiction that L is regular. Let $m > 0$ be given by the Pumping Lemma. Then let $w = (ab)^m aa(ba)^m$. Notice that $w \in L$ (with $u = (ab)^m$, $s = a$, $v = (ba)^m$) and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Since $|xy| \leq m$, we know that y is a substring of $(ab)^m$. Now let $i = 0$. Then $w_0 = xz = raa(ba)^m$ for some r with $|r| < |(ab)^m| = 2m$. I claim that $w_0 \notin L$.

To see this, suppose towards a contradiction that $w_0 \in L$. Then we can write $w_0 = uss^Rv$ with $u, v, s \in \{a, b\}^+$ and $|u| \geq |v|$. But also we know that $w_0 = raa(ba)^m$. Since $|r| < 2m$ but $|u| \geq |v|$ we must have that ra is a prefix of u. So ss^Rv is a substring of $a(ba)^m$. Now suppose the last symbol of s is a. (If the last symbol of s is b the proof is similar.) Notice then that aa is a substring of ss^R. But this is impossible because aa is not a substring of $a(ba)^m$. This contradiction proves that $w_0 \notin L$.

But this contradicts the Pumping Lemma. So L is not regular. □

(14) Let $L = \{ uu^Rv : u, v \in \{a, b\}^+ \}$. Then L is not regular.

Proof. This is a very difficult problem. It turns out that it is not possible to apply the Pumping Lemma directly to L in order to derive a contradiction. So I will use another strategy. Assume towards a contradiction that L is regular. Let r be the following regular expression: $(ab)^*(ab)(ba)(ba)^*b$. Let $L_1 = L \cap L(r)$. If L is regular then so is L_1. We will apply the Pumping Lemma to L_1 to derive a contradiciton. Notice that $L_1 = \{ (ab)^s(ba)^tb : t \geq s \geq 1 \}$. So assume that this L_1 is regular and we will derive a contradiction. Let $m > 0$ be given by the Pumping Lemma. Then let $w = (ab)^m(ba)^mb$. Notice that $w \in L_1$ and $|w| \geq m$. So let $w = xyz$ be the decomposition of w given by the Pumping Lemma. Let us consider 4 possibilities for what y looks like:

Case 1. *y starts with an a and ends with a b.*
So then $y = (ab)^k$ for some k with $1 \leq k \leq m/2$. In this case, let $i = 2$. Then $w_i = w_2 = xy^2z = (ab)^{m+k}(ba)^mb$. So $w_i \notin L$. But this contradicts the Pumping Lemma. So L_1 is not regular.

Case 2. *y starts and ends with an a.*
In this case, let $i = 2$. Then $w_i = w_2 = xyyz$. Since y starts and ends with an a, aa is a substring of yy. But it is easy to see that aa is not a substring of any string in L_1. So $w_2 \notin L_1$. But this contradicts the Pumping Lemma. So L_1 is not regular.

Case 3. *y starts and ends with an b.*
So then $y = b(ab)^k$ for some k with $0 \leq k < m/2$. Also $x = (ab)^s a$ and $z = (ab)^t (ba)^m b$ for some numbers s and t such that $s + k + t + 1 = m$. In this case let $i = 2$. Then $w_i = w_2 = xyyz = (ab)^s ab(ab)^k b(ab)^k (ab)^t (ba)^m b = (ab)^{s+1+k} b(ab)^{k+t} (ba)^m b$. Clearly $w_2 \notin L(r)$ so $w_2 \notin L_1$. But this contradicts the Pumping Lemma. So L_1 is not regular.

Case 4. *y starts with a b and ends with an a.*
So then $y = b(ab)^k a$ for some k with $0 \leq k < m/2$. Also $x = (ab)^s a$ and $z = b(ab)^t (ba)^m b$ for some numbers s and t such that $s + k + t + 2 = m$. In this case let $i = 2$. Then $w_i = w_2 = xyyz = (ab)^s ab(ab)^k ab(ab)^k ab(ab)^t (ba)^m b = (ab)^{s+1+k+1+k+1+t} (ba)^m b = (ab)^{s+k+t+3} (ba)^m b = (ab)^{m+1} (ba)^m b$. So $w_2 \notin L_1$. But this contradicts the Pumping Lemma. So L_1 is not regular.

□