Group A

Answer *ALL* questions.

A1. Suppose all the roots of the equation $x^3 + bx - 2017 = 0$ (where b is a real number) are real. Prove that exactly one root is positive.
[6]

A2. Let a, b, c and d be real numbers such that $a + b = c + d$ and $ab = cd$. Prove that $a^n + b^n = c^n + d^n$ for all positive integers n.
[8]

A3. Let $B = \{1, 2, 3, 4\}$. A set $S \subseteq B \times B$ is called a symmetric set of B if for all $x, y \in B$,

$$(x, y) \in S \Rightarrow (y, x) \in S.$$

Find the number of symmetric sets of B.
[8]

A4. Let $\lceil x \rfloor$ denote the integer nearest to x. For example, $\lceil 1.1 \rfloor = 1$, $\lceil 1.5 \rfloor = 1$ and $\lceil 1.6 \rfloor = 2$. Draw the graph of the function $y = |x - \lceil x \rfloor|$ for $0 \leq x \leq 4$. Find all the points x, $0 \leq x \leq 4$, where the function is not differentiable. Justify your answer.
[8]

Group B

Section I : Computer Science

Answer any **FIVE** questions.

C1. (a) Consider an alphabet $\Sigma = \{1,\ 2,\ 3\}$. Design a deterministic finite-state automaton (DFA) that accepts all strings in Σ^* in which the digits appear in non-decreasing sequence, from left to right. For example, the strings 1123 and 222 would be accepted, whereas 21333 will not be accepted.

(b) Show that if the edge set of a graph $G(V, E)$ with n nodes can be partitioned into 2 trees, then there is at least one vertex of degree less than 4 in G.

[7 + 7 = 14]

C2. (a) Write a complete ANSI C code using recursion to calculate the sum (s) of the digits of an integer number (i) consisting of maximum 5 digits. For example, (1) if $i = 12345$, then your program should print $s = 15$, (2) if $i = 457$, then $s = 16$.

(b) Write a C program to find all permutations of a string (having at most 6 characters). For example, a string of 3 characters like "abc" has 6 possible permutations: "abc", "acb", "bca", "bac", "cab", "cba".

[7 + 7 = 14]

C3. (a) Let $R(A, B, C)$ be a relation with primary key (A) and $S(A, D, E)$ a relation with primary key (A, D). Each of the relations has n tuples. If the number of tuples in R *natural join* S is m, then determine the number of tuples in R *natural left outer join* S.

(b) Consider the following relations:

STD_CHOICES (Student_ID, Course_ID, Semester) and
COURSE_ASSIGN (Teacher_ID, Course_ID, Semester).

The former indicates the choice of courses for students and the latter indicates the courses assigned to teachers for different semesters. Note that each student may take multiple courses, each teacher can teach multiple courses and each course can also be taught by multiple teachers.

Write the relational calculus expression to output the ID for all the students who have not been taught by the same teacher in more than one course across all semesters.

[7 + 7 = 14]

C4. A file F holds the non-zero elements of two large $n \times n$ matrices, A and B. The matrix entries are stored as triplets $(i, j, value)$, where $value$ is the $(i, j)^{\text{th}}$ element of a matrix. The file first stores the elements of A and then those of B. The matrix elements are stored in F in an arbitrary order. In each matrix, only the elements

$$\{(i, i) : i = 1, \ldots, n\},$$
$$\{(j, j+1) : j = 1, \ldots, n-1\}, \text{ and}$$
$$\{(j+1, j); j = 1, \ldots, n-1\}$$

are non-zero. You are to add A and B and store the sum in C and then print A, B and C. Due to limited memory, for storing all three matrices, you can use space to hold only up to $9n$ values (NOT triplets). Is it possible to have an $O(n)$ solution? If no, give reasons. If yes, provide a solution. Clearly explain the data structure and how you are going to store, retrieve, and add the elements.

[14]

C5. (a) An operating system contains three resource classes. The number of resource units in these classes are 7, 7 and 10 respectively. The current resource allocation state is shown below:

Process	Allocated Resources			Maximum Requirements		
	R1	R2	R3	R1	R2	R3
P_1	2	2	3	3	6	8
P_2	2	0	3	4	3	3
P_3	1	2	4	3	4	4

i. Is the current allocation state safe? Justify.

ii. If process P_1 now requests $(1, 1, 0)$ resources, then what will be the status of the new state?

(b) Consider a paging system with the page table stored in memory. If a memory reference takes 200 nanoseconds, how long does a paged memory reference take? If we add a Translation Lookaside Buffer (TLB) and 75 percent of all page-table references are TLB hits, what will then be the effective memory reference time? Assume that finding a page-table entry in the TLB takes 20 nanoseconds, if the entry is present.

[(6+4) + 4 = 14]

C6. Let $A = (a_1, a_2, \cdots, a_n)$ be an array of n distinct numbers. The array may not be sorted. The *first* element a_1 is said to be a *blip* if $a_1 > a_2$. Similarly, the *last* element a_n is said to be a *blip* if $a_n > a_{n-1}$. Among the remaining elements, an element a_i is said to be a *blip* if $a_i > a_{i-1}$ and $a_i > a_{i+1}$ where $i \in \{2, 3, \cdots, n-1\}$.

Design an $O(\log n)$ time algorithm for finding a *blip* in A. Justify the complexity of your algorithm.

[14]

C7. (a) Show that $\{1, A\bar{B}\}$ is functionally complete, i.e., any Boolean function with variables A and B can be expressed using these two primitives.

(b) Define a Boolean function $F(X_1, X_2, X_3, X_4, X_5, X_6)$ of six variables such that

$$
\begin{aligned}
F \; &= 1, \text{when three or more input variables are at logic 1.} \\
&= 0, \;\text{otherwise.}
\end{aligned}
$$

How many essential prime implicants does F have? Justify why they are essential.

[5 + 9 = 14]

C8. (a) Write the number $(-5)^{\frac{1}{2}}$ in single precision IEEE 754 floating point form.

(b) Consider a simple code $\mathcal{C}$ for error detection and correction. Each codeword in $\mathcal{C}$ consists of 2 data bits $[d_1, d_0]$ followed by 3 check bits $[c_2, c_1, c_0]$. The check bits are computed as follows: $c_2 = d_1 + d_0$, $c_1 = d_1$ and $c_0 = d_0$, where '+' is a modulo-2 addition.

i. Write down all the codewords for $\mathcal{C}$.

ii. Determine the minimum Hamming distance between any two distinct codewords of $\mathcal{C}$.

[6 + (4+4) = 14]

Section II : Engineering and Technology

Answer any **FIVE** questions.

E1. (a) How long does a hollow spherical object of radius 10 cm take, starting from rest, to roll down an inclined plane of length 6 m without slipping? The plane is inclined at an angle of $30°$ with horizontal and the acceleration due to gravity is 10 m/s^2.

(b) Consider Figure 1. A particle at rest at point A is allowed to slide along a frictionless path APB. It leaves the path at B and finally reaches the point C on the ground. Given that the tangent to the path at point B makes an angle $30°$ with horizontal and acceleration due to gravity is 10 m/s^2, calculate the horizontal distance between the points B and C.

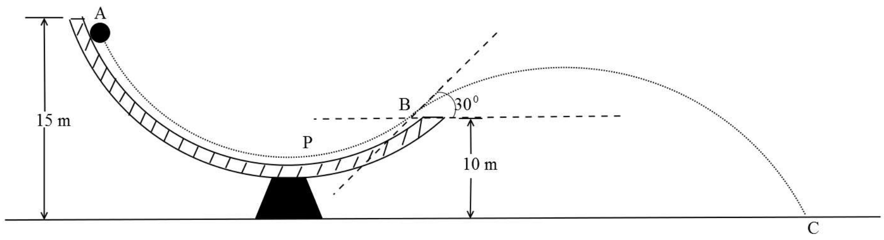

Figure 1: Figure for Question E1 (b)

[7 + 7 = 14]

E2. (a) A diatomic ideal gas is heated at constant volume until its pressure is tripled. It is again heated at constant pressure until its volume is doubled. Find the molar heat capacity of the whole process in terms of R, the universal gas constant.

(b) Three moles of an ideal mono-atomic gas undergo a cycle as shown in the P-T plane (Figure 2). The temperature of the gas in different states are $T_A = 400$ K, $T_B = 800$ K, $T_C = 2400$ K, $T_D = 1200$ K. Calculate the work done by the gas during the cycle.

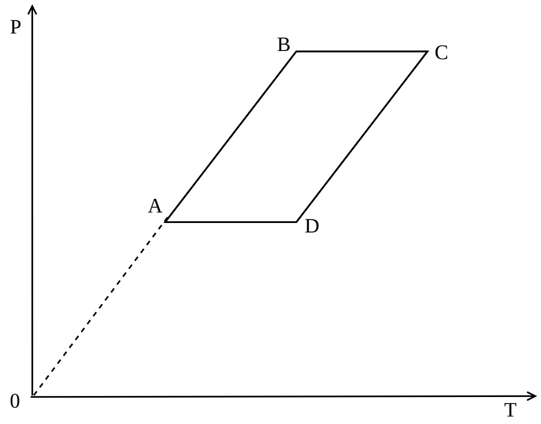

Figure 2: Figure for Question E2 (b)

[7 + 7 = 14]

E3. An infinitely long straight conductor and an isosceles triangular conductor, lying in a plane, are separated from each other as shown in Figure 3. Given $a = 10\ cm$, $b = 20\ cm$ and $h = 10\ cm$, calculate the coefficient of mutual inductance between these two conductors. [Take $\log_e 2 = 0.693$]

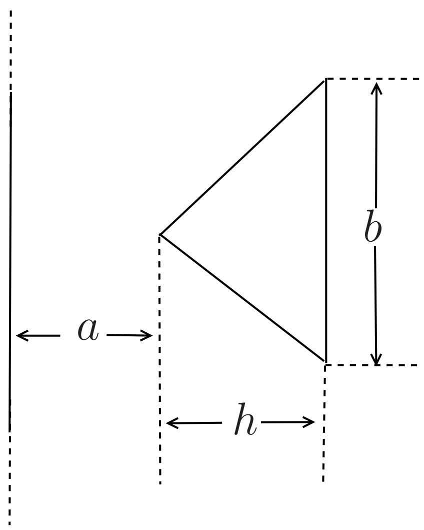

Figure 3: Figure for Question E3

[14]

E4. (a) A circuit is shown in Figure 4. Find the value of the capacitor C for which the poles of the impedance $Z(s)$ will be real and coincident. The resistance R is 100 Ω and the inductor L is 1 mH.

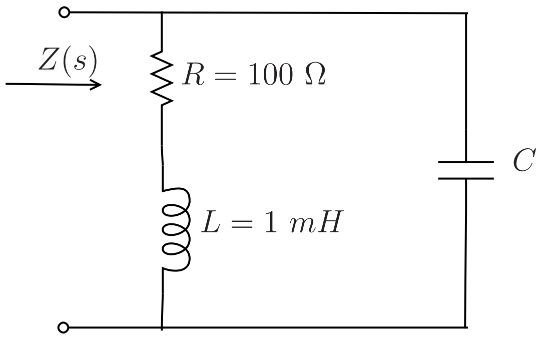

Figure 4: Circuit for Question E4 (a)

(b) An inductive circuit is connected with an alternating voltage source of RMS voltage 200 V and angular frequency $\sqrt{3} \times 10^2$ rad/s. The circuit draws an average power of 200 W at a power factor of 0.5. Determine the value of the capacitor that must be added to the circuit in series in order to increase its power factor to 1.

[6 + 8 = 14]

E5. A shunt wound 200 V, 44 kW d.c. machine has 44 Ω shunt resistance and 0.2 Ω armature resistance. The machine is used as a generator running with a speed of 530 $r.p.m.$ Calculate the torque of the machine when it is used as a motor drawing the same power. Consider a drop of 2 V at each brush.

[14]

E6. A 100 kVA transformer working at 0.96 power factor has an efficiency of 80% both at full-load and half-load. Calculate the all-day efficiency if it supplies full-load during 8 AM to 8 PM and $(\frac{3}{4})^{\text{th}}$ load from 8 PM to 8 AM.

[14]

E7. (a) Simplify the expression using Boolean postulates.

$$\overline{\overline{X\overline{Y} + XYZ} + X(Y + X\overline{Y})}$$

(b) In the combinatorial circuit shown in Figure 5, g_1 is an unknown logic gate and $g_2(A, B) = AB + \bar{A}\bar{B}$. A partial truth table of the Boolean function realized by the circuit is given in Table 1. Complete this table with justification.

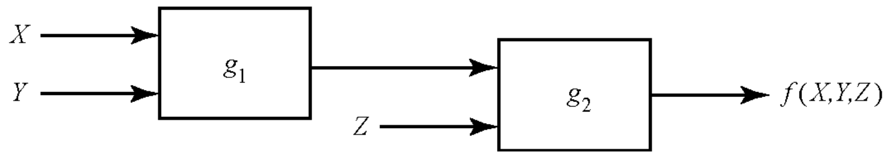

Figure 5: Figure for Question E7 (b)

Table 1: Table for Question E7 (b)

X	Y	Z	f
0	0	0	
0	1	0	
1	0	0	0
1	1	0	1
0	0	1	1
0	1	1	0
1	0	1	
1	1	1	

[5 + 9 = 14]

E8. A synchronous sequence-detector circuit is to be designed with one primary input x and one primary output y that detects three or more consecutive 1s in a bit string arriving at the input line. (a) Draw the state diagram, and (b) implement the circuit using D flip-flops and additional logic as needed. All D flip-flops are initially reset to zero.

[7 + 7 = 14]

E9. A full-wave rectifier uses two diodes and a center-tapped transformer, where r_d denotes the ON-state resistance of each diode, and r_s is the resistance of the secondary winding of the transformer. No capacitor is used for voltage regulation. An input a.c. signal $V_m \sin \omega t$ is applied to the primary side of the transformer. The output of the rectifier supplies an average current i_{dc} (d.c. component) to load R_L as shown in Figure 6. Derive the expressions for V and R_S.

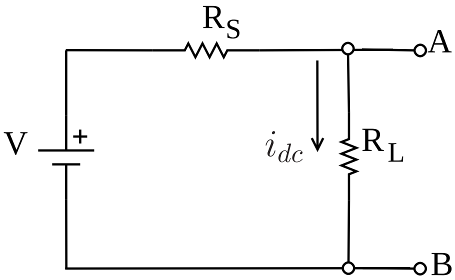

Figure 6: Circuit for Question E9

[14]

E10. Calculate the equivalent resistance R_{AB} between junction points A and B as shown in Figure 7.

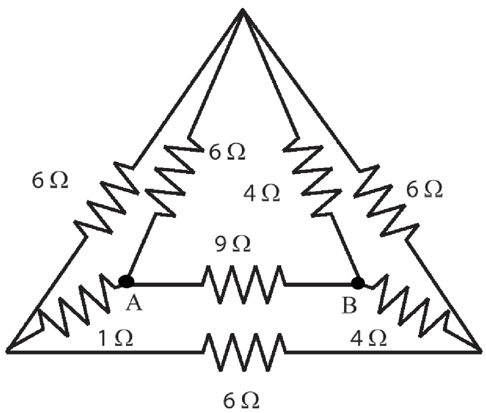

Figure 7: Circuit for Question E10

[14]

E11. (a) On execution of the following program what will be the output? Justify your answer.

```
#include <stdio.h>
#include <string.h>

void Ichange(char *String1, char *String2);
int main()
{
int i,j;
char str[7][15]={"THIS","TEST","IS",
                  "NOT","AN","EASY","ONE"};

printf("\nThe input string is : ");
for(i=0;i<=6;i++) {
    printf("%s ", str[i]);
}

for(i=0;i<6;i++)
for(j=0;j<6-i;j++)
if(strcmp(str[j+1], str[j]) < 1)
{
    Ichange(str[j], str[j+1]);
}

printf("\nAfter rearrangement : ");
for(i=0;i<=6;i++){
    printf("%s ", str[i]);
}
   return(0);
}

void Ichange(char *String1, char *String2)
{

char temp[15];
        strcpy(temp,String1);
        strcpy(String1,String2);
        strcpy(String2,temp);
}
```

(b) Find out the output of the following program when it is executed. Justify your answer.

```
#include <stdio.h>

int number(int);

int main()
{
int  VAR1=887, VAR2;

VAR2 = number(VAR1);

printf("Input=%d and Output=%d\n",VAR1,VAR2);

return 0;
}

int number(int VAR3)
{
if (VAR3 == 0)
{
    return 0;
}
else
{
    return (VAR3 % 3 +10 * number(VAR3/3));
}
}
```

[(4+3) + (4+3) = 14]

E12. (a) A body of mass 1 kg, fastened to one end of a steel wire, is rotating along a horizontal circle of radius 20 cm with a constant speed. The steel wire has a cross-sectional area 2×10^{-6} m^2 and the Young's modulus of steel is 2×10^{11} N/m^2. If the wire is elongated by 0.001 mm due to rotation of the body, calculate the linear speed of the body.

(b) A particle of mass m and a ring of mass M are released from the shown position (Figure 8) in free space. Calculate their speeds when the particle crosses the plane of the ring.

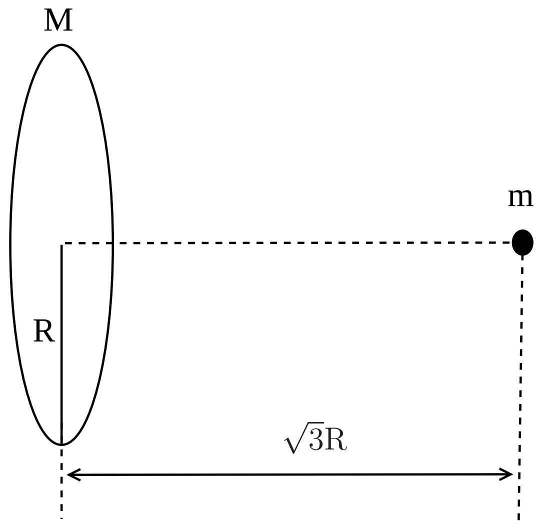

Figure 8: Figure for Question E12 (b)

[7 + 7 = 14]

Section III : Mathematics

Answer any **FIVE** questions.

M1. (a) Determine all trees whose complements are also trees.

(b) Let $\mathcal{G}$ be a connected graph, which is not a tree. Let $Subt(\mathcal{G}, e)$ be an operation on $\mathcal{G}$ which removes the edge e from $\mathcal{G}$ *only if e is an edge of $\mathcal{G}$ and is present in a cycle.* Show that for every spanning tree T of $\mathcal{G} = \mathcal{G}_0$, there exists a sequence of edges $e_1, \ldots, e_k$ such that $T = \mathcal{G}_k$, where $\mathcal{G}_i = Subt(\mathcal{G}_{i-1}, e_i)$, $1 \leq i \leq k$.

[7 + 7 = 14]

M2. Let **G** be a finite group and the only automorphism of **G** is the identity map.

(a) Show that **G** is Abelian.

(b) Prove that every non-identity element in **G** has order 2.

[7 + 7 = 14]

M3. (a) Let $\mathcal{F}$ be a finite field of characteristic 3. Show that every $y \in \mathcal{F}$ has a unique cube root in the same field, i.e., for every $y \in \mathcal{F}$ there is a unique $x \in \mathcal{F}$ such that $x^3 = y$.

(b) Let **G** be a group of cardinality 2016. Show that there exists a non-identity element x in **G** such that $x = x^{-1}$.

[7 + 7 = 14]

M4. Let $f_1 : [0, 4] \mapsto [0, 4]$ be defined by $f_1(x) = 3 - (x/2)$. Let $f_n(x) = f_1(f_{n-1}(x))$ for $n \geq 2$.

(a) Show that $\lim\limits_{n\to\infty} f_n(0)$ exists.

(b) Find the set of all x for which $\lim\limits_{n\to\infty} f_n(x)$ exists and also find the corresponding limits.

[6 + 8 = 14]

M5. Let $\mathcal{F}$ be a finite field of cardinality N and let A be an $n \times k$ $(k \leq N)$ matrix of rank r with entries from the field $\mathcal{F}$.

(a) Find the number of solutions $\mathbf{x} = (x_1, \ldots, x_k)^T \in \mathcal{F}^k$ for the system of linear equations

$$A\mathbf{x} = \mathbf{0}$$

where $\mathbf{x}^T$ is the transpose of $\mathbf{x}$ and $\mathbf{0}$ is the zero column vector of size n.

(b) Let $\mathbf{b}$ be a column vector of size n whose entries are from $\mathcal{F}$. Show that the number of solutions of $\mathbf{x} = (x_1, \ldots, x_k)^T \in \mathcal{F}^k$ for the system of linear equations

$$A\mathbf{x} = \mathbf{b}$$

is at most $\underbrace{N(N-1)\cdots(N-k+r+1)}_{(k-r)\text{ terms}}$, when x_i's are distinct.

[6 + 8 = 14]

M6. (a) Suppose π_n denotes the number of primes between 1 and n. Let there exist two real numbers A and B with $A \geq (3B/4)$ such that

$$\frac{An}{\log_e n} < \pi_n < \frac{Bn}{\log_e n}.$$

Show that for any positive integer n, there exists a prime between n and $2n$.

(b) Find all positive integer pairs of solution (x, y) such that

$$xy = 11(x + y).$$

[7 + 7 = 14]

M7. (a) Let f be a real valued function on $\mathcal{R}$. If, for all real x,

$$f(x) + 3f(1-x) = 5$$

holds, then show that f is a constant function.

(b) Solve the differential equation

$$x^2(x^2-1)\frac{dy}{dx} + x(x^2+1)y = x^2 - 1.$$

[6 + 8 = 14]

M8. Let $f : [0,1] \mapsto [0,\infty)$ be a continuous function. Let $a = \inf_{0 \le x \le 1} f(x)$ and $b = \sup_{0 \le x \le 1} f(x)$. For every positive integer m, define

$$c_m = \left[\int_0^1 (f(x))^m dx\right]^{1/m}.$$

(a) Prove that $a \le c_m \le b$ for all integers $m \ge 1$.

(b) Show that $\lim_{m\to\infty} c_m$ exists and find its value.

[6 + 8 = 14]

Section IV : Physics

Answer any **FIVE** questions.

P1. (a) A ball of mass m, attached to a light inextensible string, is rotating in a vertical circle of radius l. Its speed is v when the string is horizontal. Calculate the speed of the ball and tension in the string at the highest and lowest points of the circular path.

(b) A particle of mass m is moving in a central force field with potential energy given by $V(r) = kr^n$, where r is the distance of the particle from the center of force and k and n are constants.

i. If the particle has angular momentum J, find the radius of the circular orbit.

ii. Find the condition on n for stable circular orbit (i.e., $\frac{d^2U(r)}{dr^2} > 0$, where $U(r)$ is the effective potential).

[(3+4) + (3+4) = 14]

P2. (a) Consider a point mass m, connected to a fixed point by a spring of spring constant k and of equilibrium length l (as shown in Figure 9). The mass m is oscillating in a plane under a constant gravitational field, with g being the acceleration due to gravity.

i. Construct the Lagrangian of the system in polar coordinates (r, θ) with r being the distance of the mass from the fixed point and θ being the angle made by the spring with the vertical line.

ii. Derive the equations of motion.

iii. Write down the Hamiltonian (energy) in terms of coordinates and conjugate momenta.

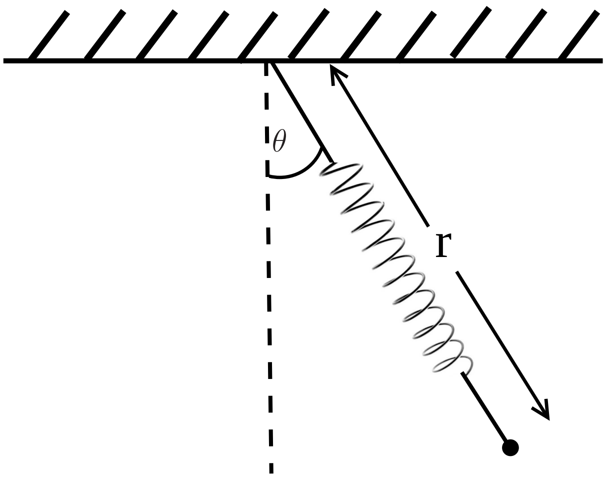

Figure 9: Figure for Question P2 (a)

(b) A muon is travelling inside your laboratory at a speed $\frac{3}{5}c$, where c is the speed of light in vacuum. How long does it last? It is given that the lifetime of muon is 2×10^{-6} seconds.

[(4+3+3) + 4 = 14]

P3. (a) For an ideal gas let the molar specific heat at constant pressure and constant volume be C_P and C_V respectively. From the first law of thermodynamics find C_P, when $C_V = \frac{5}{2}R$, where R is the universal gas constant. [No credit will be given for just stating the answer.]

(b) An ideal gas is taken from state a to state b as shown in Figure 10 using three different paths: acb, adb and ab. The pressure $P_2 = 2P_1$ and the volume $V_2 = 2V_1$. Compute the heat supplied to the gas along each of the two paths acb, adb in terms of R and T_1.

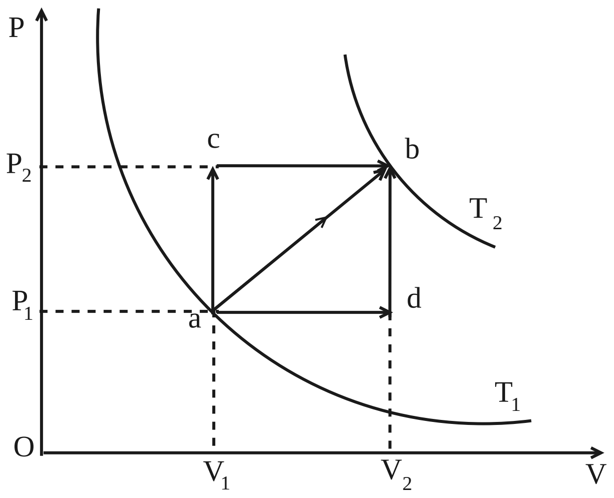

Figure 10: Figure for Question P3 (b)

(c) An ideal gas of N spinless atoms occupies a volume V at a temperature T. Each atom has only two energy levels separated by an energy Δ. Find the chemical potential, free energy and average energy.

[3 + 6 + (1+2+2) = 14]

P4. (a) Using Gauss law compute the capacitance per unit length of two infinitely long cylindrical conductors of radii a and b that are parallel and separated by a distance $d \gg a, b$, as shown in Figure 11.

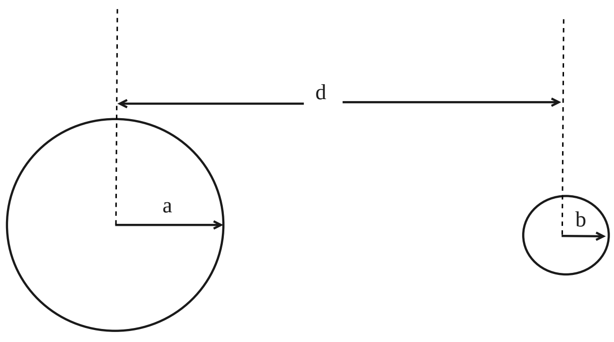

Figure 11: Figure for Question P4 (a)

(b) An electromagnetic wave with electric field given by

$$E_y = E_0 e^{i(kz-\omega t)}, \quad E_x = E_z = 0;$$

is propagating in a medium with n electrons per unit volume. Write down Maxwell's equations for the field in the medium. Ignore action of the magnetic field and interaction between the electrons.

[6 + 8 = 14]

P5. Consider a square lattice $\mathcal{L}$ (with L lattice sites in each of $\mathbf{x}$ and $\mathbf{y}$-directions) on which each lattice site is labelled by two integers (m,n) such that the lattice vector $\vec{r}_i \equiv \{x_i = ma, y_i = na\}$, with a being the distance between two neighboring lattice sites. Assume periodic boundary conditions in both directions.

(a) What is the area of the unit cell in the reciprocal lattice?

(b) Assume there are electrons moving on $\mathcal{L}$ and take $a = 1$. The Hamiltonian in Tight Binding Approximation is given by

$$\hat{H} = \sum_{x_i=0}^{x_i=L-1} \sum_{y_i=0}^{y_i=L-1} \hat{h}(\vec{r}_i),$$

where,

$$\hat{h}(\vec{r}_i) = -\frac{t}{2} \sum_{\sigma=\uparrow,\downarrow} \sum_{\vec{\delta}=\pm\hat{e}_x,\pm\hat{e}_y} (\hat{c}^\dagger_\sigma(\vec{r}_i)\hat{c}_\sigma(\vec{r}_i+\vec{\delta}) + \hat{c}_\sigma(\vec{r}_i)\hat{c}^\dagger_\sigma(\vec{r}_i+\vec{\delta})),$$

$\hat{e}_x$ and $\hat{e}_y$ being the lattice vectors.
The operator $\hat{c}_\sigma$ annihilates an electron of spin-σ on the lattice site $\vec{r}_i$. The anti-commutation rules for $\hat{c}_{\sigma_1}$ and $\hat{c}^\dagger_{\sigma_2}$ are given by

$$\{\hat{c}_{\sigma_1}(\vec{r}_i), \hat{c}^\dagger_{\sigma_2}(\vec{r}_j)\} = \delta_{\vec{r}_i,\vec{r}_j}\delta_{\sigma_1,\sigma_2},$$
$$\{\hat{c}_{\sigma_1}(\vec{r}_i), \hat{c}_{\sigma_2}(\vec{r}_j)\} = \{\hat{c}^\dagger_{\sigma_1}(\vec{r}_i), \hat{c}^\dagger_{\sigma_2}(\vec{r}_j)\} = 0.$$

By applying Fourier transformation on the above anti-commutation relations in real space lattice, find the corresponding anti-commutation relations in the reciprocal lattice.

(c) Rewrite the above Hamiltonian in the following form

$$\hat{H} = \sum_{\vec{k}} E_{\vec{k}} \sum_{\sigma=\uparrow,\downarrow} \hat{c}^{\dagger}_{\sigma}(\vec{k})\hat{c}_{\sigma}(\vec{k})$$

where $\vec{k}$ is a wave-vector in the first Brillouin zone of the reciprocal lattice, and $\hat{c}_{\sigma}(\vec{k})$ ($\hat{c}^{\dagger}_{\sigma}(\vec{k})$) is the annihilation (creation) operator for a given value of $(\sigma, \vec{k})$. Find $E_{\vec{k}}$ as a function of $\vec{k}$.

[4 + 4 + 6 = 14]

P6. (a) If the wavelength of the 1st line of the Lyman series of Hydrogen atom is 1215Å, calculate the wavelength of the second line of the series and also the wavelength in the series limit.

(b) The half-life of radium is 1590 years. In how many years will one gram of the pure element lose one centigram?

(c) Ultraviolet light of wavelength 3500Å is incident on a potassium surface. The workfunction of potassium is $2eV$. Find out the maximum energy of the emitted photoelectrons (Planck's constant $h = 6.63 \times 10^{-34}$ Joule-seconds).

[(3 + 4) + 3 + 4 = 14]

P7. Let $\hat{\psi}_l(\mathbf{x}, t)$ be an $\mathbf{x}$ and t dependent operator with an index l and $\hat{Q}$ be a Hermitian operator which does not depend explicitly on time t, such that they satisfy the following equal time commutation relation

$$[\hat{Q}, \hat{\psi}_l(\mathbf{x}, t)] = -q_l \hat{\psi}_l(\mathbf{x}, t)$$

for any l, where q_l is an integer.

(a) Find the commutation relation between $\hat{Q}$ and the Hermitian adjoint of the operator $\hat{\psi}_l(\mathbf{x}, t)$.

(b) A Hamiltonian operator $\hat{H}$ is given by

$$\hat{H} = \hat{\psi}^{\dagger}_l(\mathbf{x}, t)\hat{\psi}^{\dagger}_m(\mathbf{x}, t)\hat{\psi}_n(\mathbf{x}, t)\hat{\psi}_p(\mathbf{x}, t) + \mathbf{h.c.},$$

where **h.c.** denotes the Hermitian conjugate. Find the condition for which $\hat{Q}$ is a conserved operator.

(c) Consider a simple harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2$$

where m and ω are constants.
An annihilation operator $\hat{b}$ is defined as

$$\hat{b} \;=\; \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}} \hat{X} + \frac{i}{\sqrt{m\omega\hbar}} \hat{P} \right).$$

You may consider $\hbar = 1$ and $m = 1$.

i. Express $\hat{H}$ in terms of operators $\hat{b}$ and $\hat{b}^\dagger$.

ii. Compute $\frac{d\hat{b}}{dt}$.

[2 + 7 + (3+2) = 14]

P8. (a) In the circuit shown in Figure 12, the Zener diode has a breakdown voltage $V_z = 2V$. The power rating of the diode is 20mW. Explain whether the diode will be damaged.

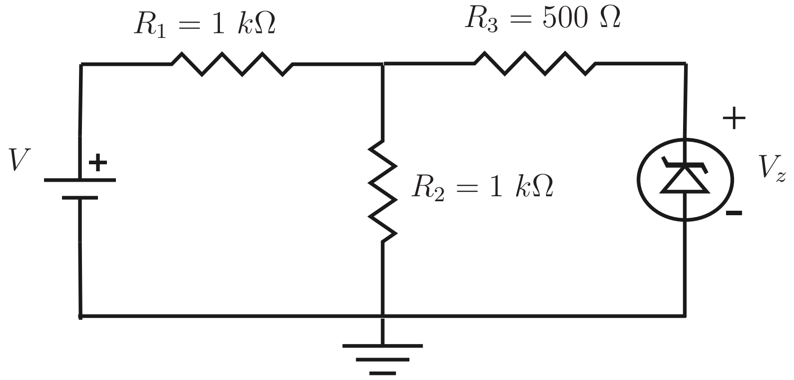

Figure 12: Circuit for Question P8 (a)

(b) The above-mentioned Zener diode is connected to two transistors Q_1, and Q_2 as shown in the circuit in Figure 13. Given that $V_{EB_1} = V_{EB_2} = 0.5V$, find R_1, R_2, and the range of R_L for which Q_1 will be in active region. Assume $\beta = 150$, for both Q_1 and Q_2.

[4 + (3+3+4) = 14]

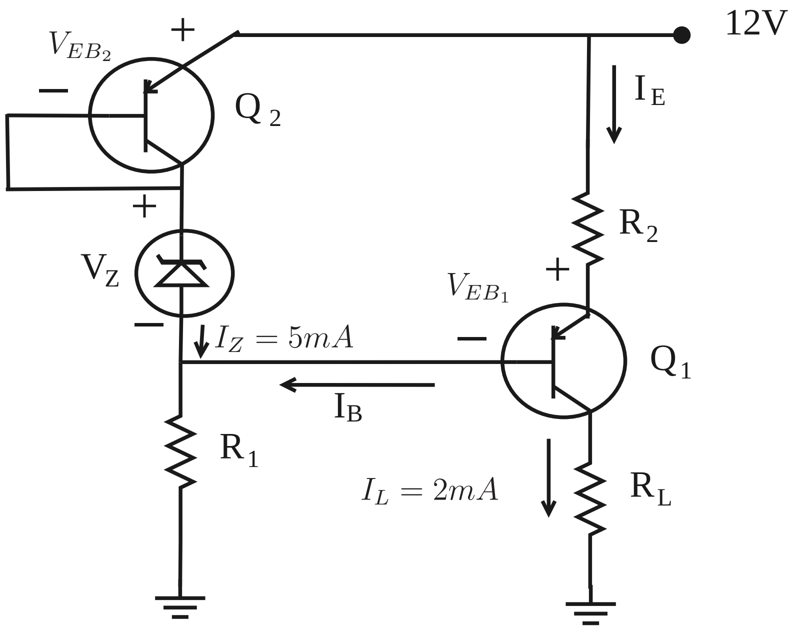

Figure 13: Circuit for Question P8 (b)

Section V : STATISTICS

Answer any **FIVE** questions.

S1. (a) Let $A = \{1, 2\}$ and $B = \{1, 2, \ldots, 10\}$. Of all the 10^2 possible maps from A to B, one is chosen at random. What is the probability that the chosen function is strictly increasing?

(b) Suppose X and Y are independent samples from the uniform distribution over the interval [0, 1]. A one meter stick is cut at X and Y meters from the left. What is the probability that the three pieces of the stick would form a triangle?

[5 + 9 = 14]

S2. Suppose X and Y are two random variables with variance 1 and correlation 0.3. Let $Z = \alpha X + \beta Y$ for some real coefficients α and β such that $|\alpha| + |\beta| = 1$.

(a) What is the minimum value of the variance of Z?

(b) Describe the set of values of α and β, for which the variance of Z attains the minimum value.

(c) How does the answer to part (b) change when the correlation between X and Y is 0.7?

(d) How does the answer to part (b) change when the correlation between X and Y is -0.3?

[6 + 3 + 2 + 3 = 14]

S3. Suppose $S_0 = 0$ and $S_k = X_1 + \cdots + X_k$ for $k \geq 1$, where $X_1, X_2, \ldots$ are independent binary random variables taking the values 0 and 1 with probabilities $1 - p$ and p, respectively. Suppose N follows Poisson distribution with mean λ (independent of $X_1, X_2, \ldots$).

(a) Show that S_N also follows a Poisson distribution.

(b) What is the conditional distribution of N given S_N? Is it also a Poisson distribution?

(c) Show that $N - S_N$ is independent of S_N.

[5 + (5 + 1) + 3 = 14]

S4. Let X be a single observation from the exponential distribution with mean μ.

(a) Find the most powerful test of size 0.1 for the null hypothesis $\mu = 1$ against the alternative hypothesis $\mu = 100$.

(b) Find the most powerful test of size 0.1 for the null hypothesis $\mu = 100$ against the alternative hypothesis $\mu = 1$.

(c) Find the range of values of X that would lead to rejection of each of the above null hypotheses.

(d) Explain how it is possible that both the null hypotheses are rejected.

(e) Show that the possibility of rejection of both the hypotheses would not arise for certain choices of the size.

[3 + 3 + 2 + 3 + 3 = 14]

S5. Suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ and $(U_1, V_1), (U_2, V_2), \ldots, (U_n, V_n)$ follow the models

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + \epsilon_i, \\ V_i &= \alpha_0 + \alpha_1 U_i + \delta_i, \end{aligned}$$

for $i = 1, \ldots, n$, where $\epsilon_1, \ldots, \epsilon_n, \delta_1, \ldots, \delta_n$ are uncorrelated errors, each with mean 0 and unknown variance σ^2.

(a) Show that all the observations can be put together in a single linear model by suitably choosing the response vector, the design matrix, the parameter vector and the error vector.

(b) Give explicit expressions for the least squares estimators of β_0, β_1, α_0 and α_1.

(c) Give an explicit expression for the usual unbiased estimator of σ^2.

(d) If it is known that $\alpha_1 = \beta_1$, then show that this restriction can be built into a different linear model with only three coefficients, and identify the design matrix of that model.

[4 + 4 + 3 + 3 = 14]

S6. Let $X_1, \ldots, X_n$ be independent random variables, each having the exponential distribution with unknown mean $\frac{1}{\lambda}$. Find the Uniformly Minimum Variance Unbiased Estimator of $P(X > 2|X > 1)$.

[14]

S7. Suppose $X_1, \ldots, X_n$ are independent samples from the normal distribution with mean μ and variance σ^2. It is known that μ cannot be negative.

(a) Find the maximum likelihood estimator (MLE) of μ.

(b) Show that the mean squared error of the MLE is less than or equal to that of the sample mean.

[9 + 5 = 14]

S8. Data were collected from an experiment with randomized block design, where there are four treatments, three blocks and one observation per cell. The total sum of squares is 2802. The F-statistics for testing the homogeneity of treatments and the homogeneity of blocks happen to be 139.64 and 15.27, respectively. Construct the ANOVA table for this experiment.

[14]