Graph Theory

Connectivity, Coloring, Matching

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[1] GATE Overflow

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Thanks to Subarna/Sukanya Das for wonderful figures

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Graph

$G = (V, E)$ is a graph and consists of a set of objects called vertices and edges such that each edge e_k is associated with an unordered pair of vertices (v_i, v_j)

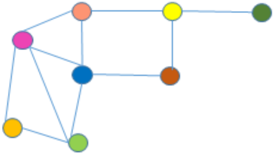

Figure: Graph

Directed Graph

A graph $G = (V, E)$ is a directed graph if each edge e_k is associated with an ORDERED pair of vertices (v_i, v_j)

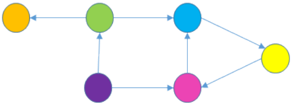

Figure: Directed Graph

Graph Types

Simple Graph

A graph that has neither self loops nor parallel edges

Figure: Simple Graphs

Graph Types

Finite Graph

By default the number of edges or vertices in a graph can be infinite. A graph with a finite number of vertices and edges is called a finite graph

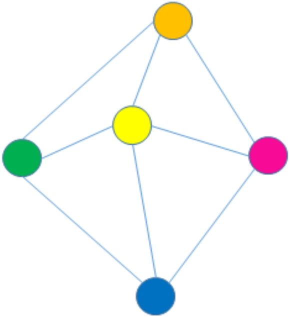

Figure: Finite Graph

Graph Types

Null graph

A graph without any edges

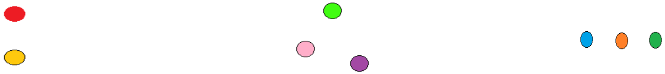

Figure: Null Graph

Incidence and Degree

Incidence

If a vertex v_i is an end vertex of an edge e_k, we say v_i and e_k are incident with each other

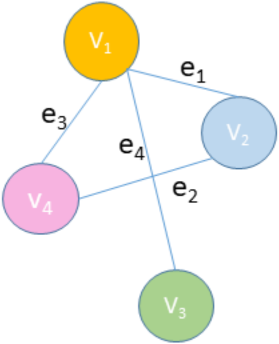

Figure: Here V_1 and e_1 are incident with each other

Incidence and Degree

Degree

The number of edges incident on a vertex v_i with self loops counted twice is called the degree of vertex v_i

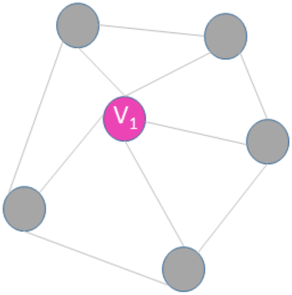

Figure: Degree of vertex V_1 is 5

Isolated vertex

A vertex having no incident edges (zero degree)

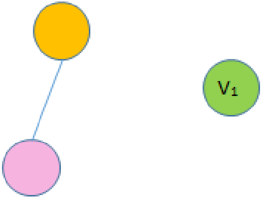

Figure: Isolated Vertex(Degree of V_1 vertex is 0)

Pendant vertex

A vertex of degree one

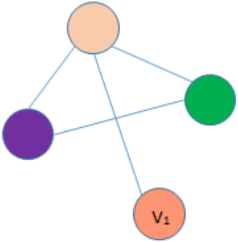

Figure: Pendant Vertex(Degree of vertex V_1 is 1)

Subgraphs

Subgraphs

A graph H is said to be a subgraph of a graph G ($H \subset G$) if all vertices and edges of H are in G and all edges of H have the same end vertices in H as in G

- Every graph is its own subgraph
- A single vertex in a graph is its subgraph
- A single edge of a graph with the end vertices is its subgraph

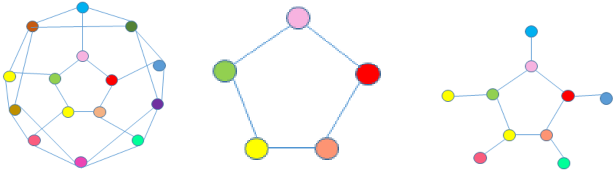

Figure: Subgraph

Complete Graph

Complete Graph

A graph in which every vertex is connected to every other vertex is called a complete graph

- Also known as a clique
- A complete graph of n vertices contain $n(n-1)/2$ edges

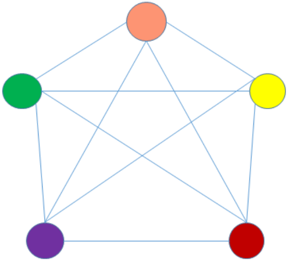

Figure: Complete Graph

Walks, Paths and Circuits

Walk

An alternating sequence of vertices and edges beginning and ending with vertices such that each edge is incident on the preceding and succeeding vertices is called a walk. A vertex can repeat in a walk but not any edge.

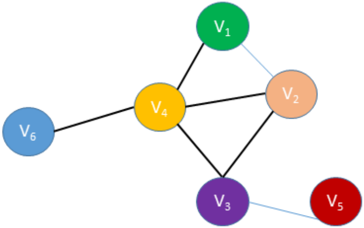

Figure: Walk : $V_6 \rightarrow V_4 \rightarrow V_2 \rightarrow V_3 \rightarrow V_4 \rightarrow V_1$
Length of the Walk : 5

Walks, Paths and Circuits

Open and Closed Walk

A walk with same start and end vertices is called an closed walk. A walk that is not closed is open walk.

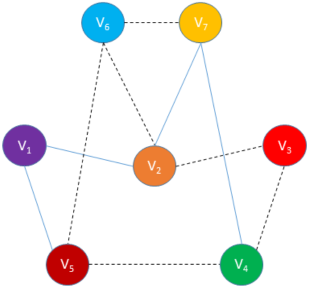

Figure: Open walk : $(V_6 \rightarrow V_2 \rightarrow V_3 \rightarrow V_4 \rightarrow V_5 \rightarrow V_6 \rightarrow V_7)$

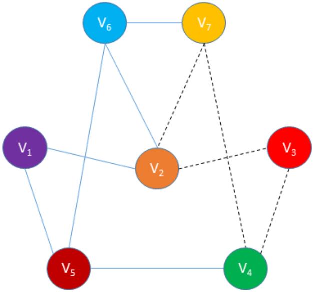

Figure: Closed walk : $(V_7 \rightarrow V_4 \rightarrow V_3 \rightarrow V_2 \rightarrow V_7)$

Walks, Paths and Circuits

Path

An open walk with no repeating vertices

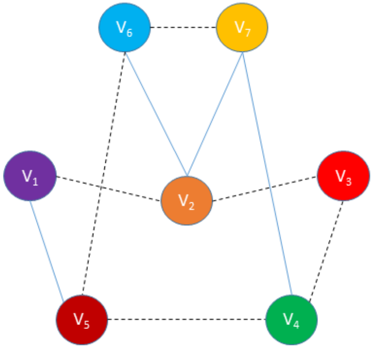

Figure: Path : $V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_4 \rightarrow V_5 \rightarrow V_6 \rightarrow V_7$

Circuit

A closed walk in which no vertex appears more than once

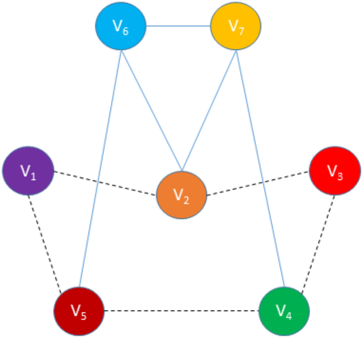

Figure: Circuit : $V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_4 \rightarrow V_5 \rightarrow V_1$

Euler Graph

Euler line and Euler Graph

A closed walk containing all edges of a graph is called an Euler line and a graph containing an Euler line is called an Euler graph

Theorem

A given connected graph G is Euler if and only if all vertices of G are of even degree. i.e.,

- A given connected graph G is Euler if all its vertices are of even degree
- If all vertices of a graph G are of even degree then G is an Euler graph

Euler Graph

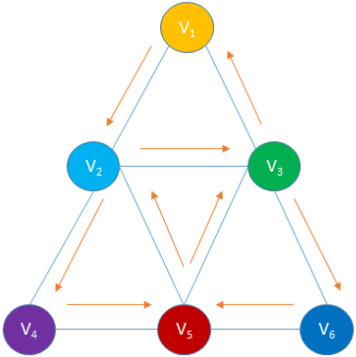

Figure: Euler Line : $V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_6 \rightarrow V_5 \rightarrow V_2 \rightarrow V_4 \rightarrow V_5 \rightarrow V_3 \rightarrow V_1$
Euler Graph

Hamiltonian Paths and Circuits

Hamiltonian Circuit

A closed walk that traverses every vertex exactly once except the starting and ending vertex. Or a circuit including every vertex of a graph. A Hamiltonian circuit in a graph of n vertices is of length n.

Hamiltonian Path

A path obtained by removing any edge from a Hamiltonian circuit. The length of a Hamiltonian path in a graph of n vertices is $n - 1$

- A graph containing a Hamiltonian circuit always has a Hamiltonian path but the reverse is not always true. i.e., some graphs have Hamiltonian path but not any Hamiltonian circuit.
- Unlike for Euler graph, there is no known necessary and sufficient condition for a graph to have a Hamiltonian circuit

Hamiltonian Paths and Circuits

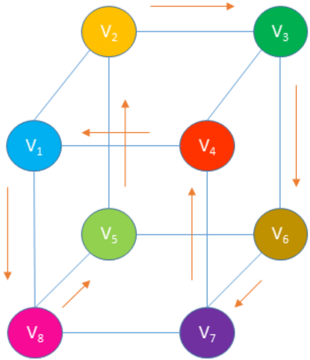

Figure: Hamiltonian Circuit : $(V_1 \rightarrow V_8 \rightarrow V_5 \rightarrow V_2 \rightarrow V_3 \rightarrow V_6 \rightarrow V_7 \rightarrow V_4 \rightarrow V_1)$

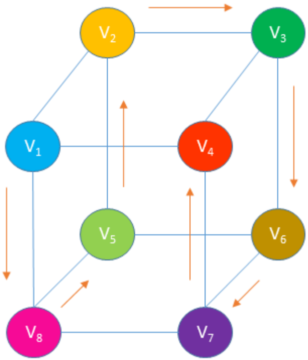

Figure: Hamiltonian Path : $(V_1 \rightarrow V_8 \rightarrow V_5 \rightarrow V_2 \rightarrow V_3 \rightarrow V_6 \rightarrow V_7 \rightarrow V_4)$

Hamiltonian Circuits

Theorem

In a complete graph of n vertices (n is odd and $n \geq 3$) there are $(n-1)/2$ edge-disjoint Hamiltonian circuits

Theorem

A sufficient (not necessary) condition for a simple graph G with n vertices to have a Hamiltonian circuit is that the degree of every vertex of G be at least $n/2$

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Havel-Hakimi Algorithm

Tells us if a given sequence of integers can form the degree sequence of a graph.

Theorem

Let $S = (d_1, \ldots, d_n)$ be a finite list of nonnegative integers that is nonincreasing. List S is graphic if and only if the finite list $S' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ has non-negative integers and is graphic.

Havel-Hakimi Algorithm

- $S = \langle 5, 5, 4, 3, 2, 2, 1 \rangle$
- Subtract 1 from the next 5 numbers after removing the leading 5

  $S' = \langle 4, 3, 2, 1, 1, 1 \rangle$

  (already in non decreasing order)
- Remove 4

  $S' = \langle 2, 1, 0, 0, 1 \rangle$
- Rearrange in non decreasing order

  $S' = \langle 2, 1, 1, 0, 0 \rangle$
- Remove 2

  $S' = \langle 0, 0, 0, 0 \rangle$
- Hence, graphic.

- $S = \langle 5, 5, 5, 3, 2, 2, 1 \rangle$
- Subtract 1 from the next 5 numbers after removing the leading 5

  $S' = \langle 4, 4, 2, 1, 1, 1 \rangle$

  (already in non decreasing order)
- Remove 4

  $S' = \langle 3, 1, 0, 0, 1 \rangle$
- Rearrange in non decreasing order

  $S' = \langle 3, 1, 1, 0, 0 \rangle$
- Remove 3

  $S' = \langle 0, 0, -1, 0 \rangle$
- Negative number came, hence, not graphic.

Connected Component

Cut-Set

Every connected subgraph of a disconnected graph G is a component of G

Tree

A tree is

- A connected graph without a circuit
- A connected graph of n vertices and $n-1$ edges
- A graph in which there is a unique path between any two vertices
- A minimally connected graph (minimally connected - removal of any one edge disconnects the graph)
- A circuit-less graph with $n-1$ edges

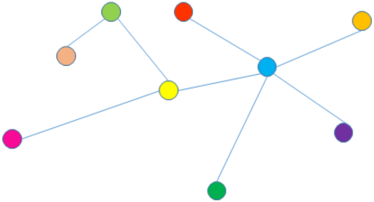


Figure: Tree

Distance and Centers

- The distance between two vertices v_i and v_j in a connected graph is the length of the shortest path between them
- Eccentricity of a vertex $E(v)$ is the distance of v with the vertex farthest from it
- A vertex with the minimal eccentricity in a graph G is called a center of G - there can be multiple centers for a graph
- A tree has either one or two centers

Counting Trees

Number of labeled trees with n vertices $(n \geq 2)$ is n^{n-2}

Spanning Trees

Spanning Tree

A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and contains all vertices of G

- An edge in a spanning tree T is called a branch of T
- An edge of a graph which is not in a given spanning tree T is called a chord of T
- A circuit formed by adding a chord to any spanning tree is called a fundamental circuit

Spanning Trees

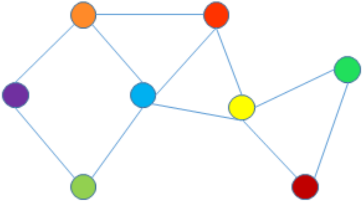
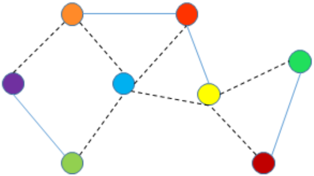


Figure: Connected Graph



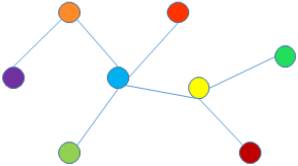


Figure: Spanning Tree

Spanning Trees

Spanning Tree

With respect to any spanning tree, a connected graph of n vertices and e edges has $n-1$ tree branches and $e-n+1$ chords

- Rank of a graph G is the number of branches in any spanning tree of G
- Nullity of a graph G (also referred to as cyclomatic number) is the number of chords with respect to any spanning tree in G
- Rank + Nullity = Number of edges

Spanning Trees

Distance between two Spanning Tree

The distance between two spanning trees T_1 and T_2 of a graph G, $d(T_1, T_2)$ is the number of edges present in one but not in the other

- We can generate a spanning tree T_2 from another spanning tree T_1 by adding a chord and removing an appropriate branch - cyclic interchange
- The minimum number of cyclic interchanges required to get a spanning tree T_2 from another spanning tree T_1 is given by $d(T_1, T_2)$
- $\max d(T_1, T_2) \leq \min(\mu, r)$, $\mu -$ nullity, $r -$ rank

Spanning Trees

Central Tree

A spanning tree with the minimal distance with any other spanning tree is called a central tree

i.e., for a central tree T_c,

$$\max_i d(T_c, T_i) \leq \max_j d(T, T_j), \forall \text{ tree } T \text{ of } G$$

Cut-Sets and Cut-Vertices

Cut-Set

A cut-set is a set of edges in a connected graph G whose removal from G leaves the graph disconnected, provided removal of no proper subset of these edges disconnects G

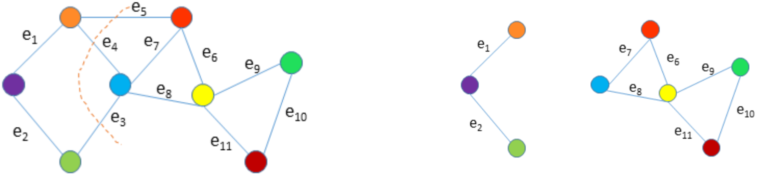


Figure: Cut set :(By removal of e_3, e_4, e_5 edges this graph will be disconnected)

Cut-Sets and Cut-Vertices

Theorem 1

Every cut-set in a connected graph G must contain at least one branch from EVERY spanning tree of G

Theorem 2 - Converse of Theorem 1

In a connected graph G every minimal set of edges containing at least one branch of EVERY spanning tree is a cut-set

Theorem 3

Every cut-set has an even number of edges in common with every circuit

Edge and Vertex Connectivity

Edge Connectivity

The number of edges in the smallest cut-set

Vertex Connectivity

The minimum number of vertices whose removal leaves the remaining graph disconnected

- The edge and vertex connectivity of a tree is one

Separable Graph

A connected graph is said to be separable if its vertex connectivity is one

Edge and Vertex Connectivity

- The edge connectivity of a graph G cannot exceed the degree of the vertex of G with the smallest degree
- The vertex connectivity of a graph G cannot exceed its edge connectivity
- The maximum vertex connectivity one can achieve with a graph of n vertices and e edges is $\left\lfloor \frac{2e}{n} \right\rfloor$

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Graph Coloring

Proper Coloring

Coloring all the vertices of a graph such that no adjacent vertices are of same color is called proper coloring of a graph

- A graph that requires minimum k different colors for proper coloring is called $k - chromatic$ graph
- Minimum number of colors required for proper coloring of a graph is called the chromatic number of the graph

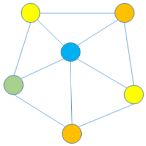

Figure: Colouring of a Graph

Chromatic Number

- A graph consisting of only isolated vertices is $1-$chromatic
- A graph with one or more edges is at least $2-$chromatic
- A complete graph of n vertices is $n-$chromatic
- A graph consisting of simply one circuit with $n \geq 3$ is

$$\begin{cases} 2\text{-}\mathit{chromatic} & \text{if } n \text{ is even} \\ 3\text{-}\mathit{chromatic} & \text{if } n \text{ is odd} \end{cases}$$

Chromatic Number

- Finding chromatic number of a graph is NP-hard - no polynomial time algorithm known so far
- Chromatic number of some specific types of graphs can be found easily
  - Every tree with 2 or more vertices is $2-$ chromatic (every $2-$ chromatic graph is not a tree)
  - A graph of at least one edge is $2-$ chromatic if and only if does not have any circuit of odd length

Chromatic Number

Figure: Chromatic Number: $\chi(G) = 1$

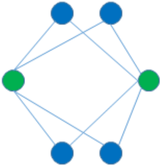

Figure: Chromatic Number: $\chi(G) = 2$

Chromatic Number

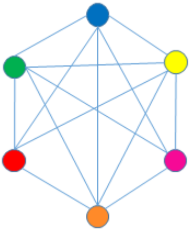

Figure: Chromatic Number: $\chi(G) = 6$

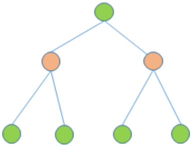

Figure: Chromatic Number: $\chi(G) = 2$

Chromatic Number

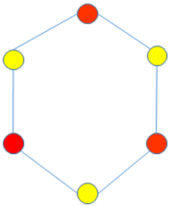

Figure: Chromatic Number: $\chi(C_6) = 2$

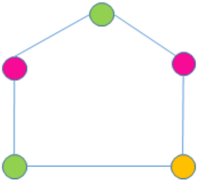

Figure: Chromatic Number: $\chi(C_5) = 3$

Chromatic Partitioning

- A proper coloring of a graph induces a partitioning of its vertices into disjoint subsets
- No two vertices in any of these partitions are adjacent

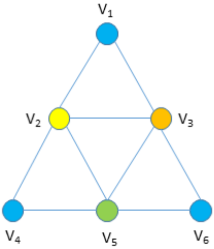

Figure: Set $A = \{V_1, V_4, V_6\}$
Set $B = \{V_2\}$
Set $C = \{V_3\}$
Set $D = \{V_5\}$
Chromatic Number: $\chi(G) = 4$

Bipartite Graph

Bipartite Graph

A graph G is called a bipartite graph if the vertex set of G can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2.

- Every $2-$chromatic graph is bipartite
- Every bipartite graph except one with two or more isolated vertices and no edges, is $2-$ chromatic

Bipartite Graph

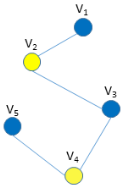

Figure: Set $A = \{V_1, V_3, V_5\}$
Set $B = \{V_2, V_4\}$

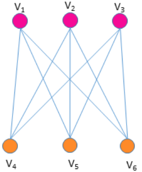

Figure: Set $A = \{V_1, V_2, V_3\}$
Set $B = \{V_4, V_5, V_6\}$

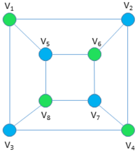

Figure: Set $A = \{V_1, V_6, V_8, V_4\}$
Set $B = \{V_2, V_3, V_5, V_7\}$

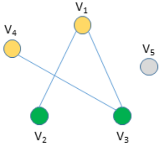

Figure: Set $A = \{V_1, V_4\}$
Set $B = \{V_2, V_3\}$

Independent Set

Independent Set

A set of vertices in a graph is said to be independent set if no two vertices in the set are adjacent

- A maximal independent set is an independent set to which no vertex can be added without losing the independence property
- The number of vertices in the largest independent set of a graph is called its independence number $\beta(G)$. If n is the number of vertices and k the chromatic number

$$\beta(G) \geq \frac{n}{k}$$

- The minimum number of maximal independent sets which collectively include all the vertices of a graph, gives its chromatic number

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Matching

Matching

A matching in a graph is a subset of its edges such that no two edges are adjacent

- A maximal matching is a matching to which no more edges can be added
- In a complete graph of 3 vertices each edge is a maximal matching
- A maximal matching with the largest number of edges is called a largest maximal matching
- The number of edges in the largest maximal matching of a graph is called its matching number

Complete Matching

Complete Matching

A matching in a bipartite graph with vertex partition V_1 and V_2 is a complete matching of vertices in V_1 into those in V_2 if there is an edge incident on each vertex of V_1

- A complete matching if it exists is a largest maximal matching
- A largest maximal matching need not be complete

Theorem

A complete matching of V_1 into V_2 in a bipartite graph exists if and only if every subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2 for all possible values of r.

Complete Matching

Complete Matching

A matching in a bipartite graph with vertex partition V_1 and V_2 is a complete matching of vertices in V_1 into those in V_2 if there is an edge incident on each vertex of V_1

- A complete matching if it exists is a largest maximal matching
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Theorem

A complete matching of V_1 into V_2 in a bipartite graph exists if and only if every subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2 for all possible values of r

Complete Matching

Theorem

In a bipartite graph a complete matching of V_1 into V_2 exists if there is a positive integer m such that

$$\text{degree of every vertex in } V_1 \geq m \geq \text{degree of every vertex in } V_2$$