Ch. 1; Q. 29:
Consider the Nim position with piles of sizes 22,19,14,11. Is it balanced (Nim sum 0)? Suppose the first player moves by taking 6 coins from the pile of size 19. How should the second player respond?
In binary, our piles are of sizes:

```
10110
10011
01110
01011
```

the Nim sum of these 4 numbers is $00000 = 0$, so this position is "balanced" and a second-player win.

The first player takes 6 from the pile of size 19, leaving us with

```
10110
01101
01110
01011
```

which has nim sum $b := 11110$.

According to our general winning strategy, we should move in a pile with a leading 1 in the same column as b, which can only be the first pile with 10110 coins, so as to leave it with $10110 \oplus 11110 = 01000$ coins.

So to win, we should take (in decimal) 13 coins from the pile with 21 coins, leaving 8.

Ch. 2; Q. 8:
How many ways can 6 men and 6 women be sat around a round table if the men and women are to alternate?
There are many ways to solve this; here's one.

Consider first the men alone.
There are $6!/6 = 5!$ circular permutations of the men.
Similarly, there are 5! circular permutations of the 6 women.

Now a circular seating of the 12 which alternates men and women corresponds to an interleaving of a circular permutation of the men with a circular permutation of the women;
given a pair of such circular permutations, to interleave them we have to choose a relative orientation of the two.
There are 6 possibilities for this.

So the answer is
$$5! * 5! * 6 = 86400$$

Ch. 2; Q. 11:

How many sets of three integers between $\{1, 2, ..., 20\}$ contain no two consecutive integers?

Let's use the Subtraction Principle, and first count the number of sets which fail the condition,

i.e. the number of subsets of $\{1, 2, ..., 20\}$ of size 3 which contain a consecutive pair of integers.

Any such set can be written uniquely in the form $\{x, x + 1, y\}$,
where $x \in \{1, ..., 19\}$ and $y \in \{1, ..., 20\}$ with $y \notin \{x, x + 1, x + 2\}$.
(If we allowed y=x+2, we would count sets of the form $\{x, x + 1, x + 2\}$ twice.) For $x < 19$, there are 17 such y, and for $x = 19$ there are 18 such y.

So there are $18 * 17 + 18 = 18^2$ possibilities.

Now the number of subsets of $\{1, ..., 20\}$ of size 3 is $\binom{20}{3}$.

So the answer to the question is

$$\binom{20}{3} - 18^2 = 1956$$

Ch. 2; Q. 18:

How many ways can 2 red and 4 blue rooks be placed on an 8x8 board such that no two attack each other?

We can reduce this to a problem close to one we solved in class by adding another two rooks of a new colour, say green.

As we did in class, the number of non-attacking configurations of 8 rooks with these colours is the number of configurations without worrying about the colours,

which correspond to permutations of $\{1, ..., 8\}$ so are $8!$ in number, multiplied by the number of ways of colouring them with 2 red, 4 blue and 2 green,

which is the number of permutations of the multiset

$$\{2 * R, 4 * B, 2 * G\},$$

which is

$$8!/(2! * 4! * 2!).$$

Now consider removing the green rooks. Each non-attacking configuration of the 6 rooks arises is obtained from precisely two non-attacking configurations of the 8 rooks.

So the answer is

$$8!(8!/(2! * 4! * 2!))/2 = 8467200.$$

Ch. 2; Q. 21:

How many permutations (anagrams) are there of the letters of the word "ADDRESSES"? How many 8-permutations?

Finding the number of permutations is a problem we solved directly in class.
We want the number of permutations of the multiset
$\{1 * A, 2 * D, 2 * E, 1 * R, 3 * S\}$ which is
$9!/(1! * 2! * 2! * 1! * 3!) = 15120$.

For the 8-permutations: an 8-permutation is a permutation of a submultiset of size 8, so we can solve this by adding up the number of permutations of each such submultiset which arises. Those submultisets are:

$\{2 * D, 2 * E, 1 * R, 3 * S\}$
$\{1 * A, 1 * D, 2 * E, 1 * R, 3 * S\}$
$\{1 * A, 2 * D, 1 * E, 1 * R, 3 * S\}$
$\{1 * A, 2 * D, 2 * E, 3 * S\}$
$\{1 * A, 2 * D, 2 * E, 1 * R, 2 * S\}$

So the answer is

$8!/(2! * 2! * 3!)$
$+8!/(2! * 3!)$
$+8!/(2! * 3!)$
$+8!/(2! * 2! * 3!)$
$+8!/(2! * 2! * 2!)$
$= 15120$.

We note that we got the same answer, which suggests we may have been missing an easier solution!

Indeed, if we think about it,
we see that the map from 9-permutations of "ADDRESSES" to 8-permutations obtained by forgetting the last letter in the permutation is a bijection,
since the last letter is determined by the rest.
So there are as many 8-permutations as 9-permutations!

Ch. 2; Q. 26:
A group of nm people are to be arranged into m teams each of n players. How many ways can this be done (a) if each team has a different name; (b) if the teams are anonymous?

(a) Let's use the division principle.
Consider arranging the nm players in a row.
Then we put the first n players in the first team, the next n in the second team, and so on.

There are $(nm)!$ arrangements of the players.
For each choice of teams,
the number of arrangements of players which result in that choice is the number of ways of permuting the members of the teams,
which is n! for each team, so $(n!)^m$.

So the answer is
$(nm)!/(n!)^m$

(b) We can obtain a choice of anonymous teams from a choice of named teams by forgetting the names.
The number of choices of named teams resulting in the same choice of anonymous teams is the number of ways of permuting the names, m!.

So the answer is
$(nm)!/((n!)^m m!)$

Ch. 2; Q. 37:
A bakery sells 6 types of pastry. It has at least a dozen of each kind; how many options are there for a dozen pastries? What if we want at least 1 of each type?
Let's assume, despite the involvement of bakers, that a dozen is 12.

The first question is one we solved directly in class: we want the number of 12-combinations of a multiset with 6 kinds and at least 12 of each kind; that number is

$$\binom{12+6-1}{12} = \binom{17}{12}.$$

If we want at least 1 of each type, then only another 6 will fit in our box, and we just have to consider the possibilities for these. But that's another instance of the same problem, and the answer is

$$\binom{6+6-1}{6} = \binom{11}{6}.$$

Ch. 2; Q. 58:
What's the probability that a 5-card poker hand contains cards of 5 different ranks, but is not a flush or a straight?
We should first work out how many hands have 5 distinct ranks.

There are 13 ranks, so the number of choices of ranks for such a hand is $\binom{13}{5}$. There are four cards of each rank (one for each suit), so for a given choice of distinct ranks there are 4^5 hands with those ranks.

So there are
$\binom{13}{5} * 4^5$ hands with 5 distinct ranks.

Now we should work out how many of these are flushes or straights.

Given a choice of ranks, there are just 4 ways it can be a flush, one for each of the suits. So there are $\binom{13}{5} * 4$ flushes.

The number of choices of 5 ranks which result in a straight is the number of possibilities for the top card in the straight, which can be anything between 5 and Ace, so 10 possibilities. For each such we have as before 4^5 choices of suits, so there are $10 * 4^5$ straights.

We have to be careful, however, not to double-count. Some hands are **both** straights **and** flushes (they're called "straight flushes" in poker), and we shouldn't count them twice. Thinking as above, there are $10 * 4$ straight flushes.

So the number of hands with 5 distinct ranks which are flushes or straights is the number which are flushes plus the number which are straights *minus* the number which are straight flushes

$$\binom{13}{5} * 4 + 10 * 4^5 - 10 * 4.$$

Now each hand is equally likely, so the probability we wanted is the number of 5-rank hands which are *not* straights or flushes divided by the number of 5-rank hands, so our final answer is

$$\left(\binom{13}{5} * 4^5 - \left(\binom{13}{5} * 4 + 10 * 4^5 - 10 * 4\right)\right) / \binom{13}{5} * 4^5$$
$$= 108545/109824$$
$$= 0.9883540938228438$$