Chapter 17

Generating Functions

Generating Functions are one of the most surprising and useful inventions in Discrete Math. Roughly speaking, generating functions transform problems about *sequences* into problems about *functions*. This is great because we've got piles of mathematical machinery for manipulating functions. Thanks to generating functions, we can apply all that machinery to problems about sequences. In this way, we can use generating functions to solve all sorts of counting problems. There is a huge chunk of mathematics concerning generating functions, so we will only get a taste of the subject.

In this chapter, we'll put sequences in angle brackets to more clearly distinguish them from the many other mathematical expressions floating around.

The *ordinary generating function* for $\langle g_0, g_1, g_2, g_3 \ldots \rangle$ is the power series:

$$G(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots .$$

There are a few other kinds of generating functions in common use, but ordinary generating functions are enough to illustrate the power of the idea, so we'll stick to them. So from now on *generating function* will mean the ordinary kind.

A generating function is a "formal" power series in the sense that we usually regard x as a placeholder rather than a number. Only in rare cases will we actually evaluate a generating function by letting x take a real number value, so we generally ignore the issue of convergence.

Throughout this chapter, we'll indicate the correspondence between a sequence and its generating function with a double-sided arrow as follows:

$$\langle g_0, g_1, g_2, g_3, \ldots \rangle \longleftrightarrow g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots$$

For example, here are some sequences and their generating functions:

$$\begin{aligned}
\langle 0, 0, 0, 0, \ldots \rangle &\longleftrightarrow 0 + 0x + 0x^2 + 0x^3 + \cdots = 0 \\
\langle 1, 0, 0, 0, \ldots \rangle &\longleftrightarrow 1 + 0x + 0x^2 + 0x^3 + \cdots = 1 \\
\langle 3, 2, 1, 0, \ldots \rangle &\longleftrightarrow 3 + 2x + 1x^2 + 0x^3 + \cdots = 3 + 2x + x^2
\end{aligned}$$

The pattern here is simple: the ith term in the sequence (indexing from 0) is the coefficient of x^i in the generating function.

Recall that the sum of an infinite geometric series is:

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

This equation does not hold when $|z| \geq 1$, but as remarked, we don't worry about convergence issues. This formula gives closed form generating functions for a whole range of sequences. For example:

$$\begin{aligned}
\langle 1, 1, 1, 1, \ldots \rangle &\longleftrightarrow 1 + x + x^2 + x^3 + \cdots &= \frac{1}{1-x} \\
\langle 1, -1, 1, -1, \ldots \rangle &\longleftrightarrow 1 - x + x^2 - x^3 + x^4 - \cdots &= \frac{1}{1+x} \\
\langle 1, a, a^2, a^3, \ldots \rangle &\longleftrightarrow 1 + ax + a^2x^2 + a^3x^3 + \cdots &= \frac{1}{1-ax} \\
\langle 1, 0, 1, 0, 1, 0, \ldots \rangle &\longleftrightarrow 1 + x^2 + x^4 + x^6 + \cdots &= \frac{1}{1-x^2}
\end{aligned}$$

17.1 Operations on Generating Functions

The magic of generating functions is that we can carry out all sorts of manipulations on sequences by performing mathematical operations on their associated generating functions. Let's experiment with various operations and characterize their effects in terms of sequences.

17.1.1 Scaling

Multiplying a generating function by a constant scales every term in the associated sequence by the same constant. For example, we noted above that:

$$\langle 1, 0, 1, 0, 1, 0, \ldots \rangle \longleftrightarrow 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

Multiplying the generating function by 2 gives

$$\frac{2}{1-x^2} = 2 + 2x^2 + 2x^4 + 2x^6 + \cdots$$

which generates the sequence:

$$\langle 2, 0, 2, 0, 2, 0, \ldots \rangle$$

Rule 11 (Scaling Rule). *If*

$$\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(x),$$

then

$$\langle cf_0, cf_1, cf_2, \ldots \rangle \longleftrightarrow c \cdot F(x).$$

The idea behind this rule is that:

$$\begin{aligned} \langle cf_0, cf_1, cf_2, \ldots \rangle &\longleftrightarrow cf_0 + cf_1 x + cf_2 x^2 + \cdots \\ &= c \cdot (f_0 + f_1 x + f_2 x^2 + \cdots) \\ &= cF(x) \end{aligned}$$

17.1.2 Addition

Adding generating functions corresponds to adding the two sequences term by term. For example, adding two of our earlier examples gives:

$$\begin{array}{rcccccccccc} & \langle & 1, & 1, & 1, & 1, & 1, & 1, & \ldots & \rangle & \longleftrightarrow \dfrac{1}{1-x} \\ + & \langle & 1, & -1, & 1, & -1, & 1, & -1, & \ldots & \rangle & \longleftrightarrow \dfrac{1}{1+x} \\ \hline & \langle & 2, & 0, & 2, & 0, & 2, & 0, & \ldots & \rangle & \longleftrightarrow \dfrac{1}{1-x} + \dfrac{1}{1+x} \end{array}$$

We've now derived two different expressions that both generate the sequence $\langle 2, 0, 2, 0, \ldots \rangle$. They are, of course, equal:

$$\frac{1}{1-x} + \frac{1}{1+x} = \frac{(1+x) + (1-x)}{(1-x)(1+x)} = \frac{2}{1-x^2}$$

Rule 12 (Addition Rule). *If*

$$\begin{aligned} \langle f_0, f_1, f_2, \ldots \rangle &\longleftrightarrow F(x), \qquad \textit{and} \\ \langle g_0, g_1, g_2, \ldots \rangle &\longleftrightarrow G(x), \end{aligned}$$

then

$$\langle f_0 + g_0,\ f_1 + g_1,\ f_2 + g_2,\ \ldots \rangle \longleftrightarrow F(x) + G(x).$$

The idea behind this rule is that:

$$\begin{aligned} \langle f_0 + g_0,\ f_1 + g_1,\ f_2 + g_2,\ \ldots \rangle &\longleftrightarrow \sum_{n=0}^{\infty} (f_n + g_n) x^n \\ &= \left(\sum_{n=0}^{\infty} f_n x^n \right) + \left(\sum_{n=0}^{\infty} g_n x^n \right) \\ &= F(x) + G(x) \end{aligned}$$

17.1.3 Right Shifting

Let's start over again with a simple sequence and its generating function:

$$\langle 1, 1, 1, 1, \ldots \rangle \longleftrightarrow \frac{1}{1-x}$$

Now let's *right-shift* the sequence by adding k leading zeros:

$$\begin{aligned}\langle \underbrace{0, 0, \ldots, 0}_{k \text{ zeroes}}, 1, 1, 1, \ldots \rangle &\longleftrightarrow x^k + x^{k+1} + x^{k+2} + x^{k+3} + \cdots \\ &= x^k \cdot (1 + x + x^2 + x^3 + \cdots) \\ &= \frac{x^k}{1-x}\end{aligned}$$

Evidently, adding k leading zeros to the sequence corresponds to multiplying the generating function by x^k. This holds true in general.

Rule 13 (Right-Shift Rule). *If* $\langle f_0, f_1, f_2, \ldots \rangle \longleftrightarrow F(x)$*, then:*

$$\langle \underbrace{0, 0, \ldots, 0}_{k \text{ zeroes}}, f_0, f_1, f_2, \ldots \rangle \longleftrightarrow x^k \cdot F(x)$$

The idea behind this rule is that:

$$\begin{aligned}\langle \overbrace{0, 0, \ldots, 0}^{k \text{ zeroes}}, f_0, f_1, f_2, \ldots \rangle &\longleftrightarrow f_0 x^k + f_1 x^{k+1} + f_2 x^{k+2} + \cdots \\ &= x^k \cdot (f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots) \\ &= x^k \cdot F(x)\end{aligned}$$

17.1.4 Differentiation

What happens if we take the *derivative* of a generating function? As an example, let's differentiate the now-familiar generating function for an infinite sequence of 1's.

$$\begin{aligned}\frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \cdots) &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ 1 + 2x + 3x^2 + 4x^3 + \cdots &= \frac{1}{(1-x)^2} \qquad (17.1) \\ \langle 1, 2, 3, 4, \ldots \rangle &\longleftrightarrow \frac{1}{(1-x)^2}\end{aligned}$$

We found a generating function for the sequence $\langle 1, 2, 3, 4, \ldots \rangle$ of positive integers!

In general, differentiating a generating function has two effects on the corresponding sequence: each term is multiplied by its index and the entire sequence is shifted left one place.

Rule 14 (Derivative Rule)**.** *If*

$$\langle f_0, f_1, f_2, f_3, \ldots \rangle \longleftrightarrow F(x),$$

then

$$\langle f_1, 2f_2, 3f_3, \ldots \rangle \longleftrightarrow F'(x).$$

The idea behind this rule is that:

$$\begin{aligned}\langle f_1, 2f_2, 3f_3, \ldots \rangle &\longleftrightarrow f_1 + 2f_2x + 3f_3x^2 + \cdots \\ &= \frac{d}{dx}\left(f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots\right) \\ &= \frac{d}{dx}F(x)\end{aligned}$$

The Derivative Rule is very useful. In fact, there is frequent, independent need for each of differentiation's two effects, multiplying terms by their index and left-shifting one place. Typically, we want just one effect and must somehow cancel out the other. For example, let's try to find the generating function for the sequence of squares, $\langle 0, 1, 4, 9, 16, \ldots \rangle$. If we could start with the sequence $\langle 1, 1, 1, 1, \ldots \rangle$ and multiply each term by its index two times, then we'd have the desired result:

$$\langle 0 \cdot 0, 1 \cdot 1, 2 \cdot 2, 3 \cdot 3, \ldots \rangle = \langle 0, 1, 4, 9, \ldots \rangle$$

A challenge is that differentiation not only multiplies each term by its index, but also shifts the whole sequence left one place. However, the Right-Shift Rule 13 tells how to cancel out this unwanted left-shift: multiply the generating function by x.

Our procedure, therefore, is to begin with the generating function for $\langle 1, 1, 1, 1, \ldots \rangle$, differentiate, multiply by x, and then differentiate and multiply by x once more.

$$\begin{aligned}\langle 1, 1, 1, 1, \ldots \rangle &\longleftrightarrow \frac{1}{1-x} \\ \langle 1, 2, 3, 4, \ldots \rangle &\longleftrightarrow \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} \\ \langle 0, 1, 2, 3, \ldots \rangle &\longleftrightarrow x \cdot \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2} \\ \langle 1, 4, 9, 16, \ldots \rangle &\longleftrightarrow \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3} \\ \langle 0, 1, 4, 9, \ldots \rangle &\longleftrightarrow x \cdot \frac{1+x}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}\end{aligned}$$

Thus, the generating function for squares is:

$$\frac{x(1+x)}{(1-x)^3} \tag{17.2}$$

17.1.5 Products

Rule 15 (Product Rule). *If*

$$\langle a_0, a_1, a_2, \dots \rangle \longleftrightarrow A(x), \quad \textit{and} \quad \langle b_0, b_1, b_2, \dots \rangle \longleftrightarrow B(x),$$

then

$$\langle c_0, c_1, c_2, \dots \rangle \longleftrightarrow A(x) \cdot B(x),$$

where

$$c_n ::= a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0.$$

To understand this rule, let

$$C(x) ::= A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We can evaluate the product $A(x) \cdot B(x)$ by using a table to identify all the cross-terms from the product of the sums:

	b_0x^0	b_1x^1	b_2x^2	b_3x^3	$\dots$
a_0x^0	$a_0b_0x^0$	$a_0b_1x^1$	$a_0b_2x^2$	$a_0b_3x^3$	$\dots$
a_1x^1	$a_1b_0x^1$	$a_1b_1x^2$	$a_1b_2x^3$	$\dots$	
a_2x^2	$a_2b_0x^2$	$a_2b_1x^3$	$\dots$		
a_3x^3	$a_3b_0x^3$	$\dots$			
$\vdots$	$\dots$				

Notice that all terms involving the same power of x lie on a /-sloped diagonal. Collecting these terms together, we find that the coefficient of x^n in the product is the sum of all the terms on the $(n+1)$st diagonal, namely,

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0. \tag{17.3}$$

This expression (17.3) may be familiar from a signal processing course; the sequence $\langle c_0, c_1, c_2, \dots \rangle$ is called the *convolution* of sequences $\langle a_0, a_1, a_2, \dots \rangle$ and $\langle b_0, b_1, b_2, \dots \rangle$.

17.2 The Fibonacci Sequence

Sometimes we can find nice generating functions for more complicated sequences. For example, here is a generating function for the Fibonacci numbers:

$$\langle 0, 1, 1, 2, 3, 5, 8, 13, 21, \dots \rangle \longleftrightarrow \frac{x}{1 - x - x^2}$$

The Fibonacci numbers may seem like a fairly nasty bunch, but the generating function is simple!

We're going to derive this generating function and then use it to find a closed form for the nth Fibonacci number. The techniques we'll use are applicable to a large class of recurrence equations.

17.2.1 Finding a Generating Function

Let's begin by recalling the definition of the Fibonacci numbers:

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \qquad \text{(for } n \geq 2\text{)} \end{aligned}$$

We can expand the final clause into an infinite sequence of equations. Thus, the Fibonacci numbers are defined by:

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= f_1 + f_0 \\ f_3 &= f_2 + f_1 \\ f_4 &= f_3 + f_2 \\ &\vdots \end{aligned}$$

Now the overall plan is to *define* a function $F(x)$ that generates the sequence on the left side of the equality symbols, which are the Fibonacci numbers. Then we *derive* a function that generates the sequence on the right side. Finally, we equate the two and solve for $F(x)$. Let's try this. First, we define:

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$

Now we need to derive a generating function for the sequence:

$$\langle 0,\ 1,\ f_1 + f_0,\ f_2 + f_1,\ f_3 + f_2,\ \ldots \rangle$$

One approach is to break this into a sum of three sequences for which we know generating functions and then apply the Addition Rule:

$$\begin{array}{rlllllllcl} & \langle & 0, & 1, & 0, & 0, & 0, & \ldots & \rangle & \longleftrightarrow & x \\ & \langle & 0, & f_0, & f_1, & f_2, & f_3, & \ldots & \rangle & \longleftrightarrow & xF(x) \\ + & \langle & 0, & 0, & f_0, & f_1, & f_2, & \ldots & \rangle & \longleftrightarrow & x^2F(x) \\ \hline & \langle & 0, & 1 + f_0, & f_1 + f_0, & f_2 + f_1, & f_3 + f_2, & \ldots & \rangle & \longleftrightarrow & x + xF(x) + x^2F(x) \end{array}$$

This sequence is almost identical to the right sides of the Fibonacci equations. The one blemish is that the second term is $1 + f_0$ instead of simply 1. However, this amounts to nothing, since $f_0 = 0$ anyway.

Now if we equate $F(x)$ with the new function $x + xF(x) + x^2F(x)$, then we're implicitly writing down *all* of the equations that define the Fibonacci numbers in one fell swoop:

$$\begin{array}{rcccccccc} F(x) & = & f_0 + & f_1 & x + & f_2 & x^2 + & f_3 & x^3 + \cdots \\ \shortparallel & & \shortparallel & \shortparallel & & \shortparallel & & \shortparallel & \\ x + xF(x) + x^2F(x) & = & 0 & + (1 + f_0) & x + & (f_1 + f_0) & x^2 + & (f_2 + f_1) & x^3 + \cdots \end{array}$$

Solving for $F(x)$ gives the generating function for the Fibonacci sequence:

$$F(x) = x + xF(x) + x^2F(x)$$

so

$$F(x) = \frac{x}{1 - x - x^2}.$$

Sure enough, this is the simple generating function we claimed at the outset.

17.2.2 Finding a Closed Form

Why should one care about the generating function for a sequence? There are several answers, but here is one: if we can find a generating function for a sequence, then we can often find a closed form for the nth coefficient— which can be pretty useful! For example, a closed form for the coefficient of x^n in the power series for $x/(1 - x - x^2)$ would be an explicit formula for the nth Fibonacci number.

So our next task is to extract coefficients from a generating function. There are several approaches. For a generating function that is a ratio of polynomials, we can use the method of *partial fractions*, which you learned in calculus. Just as the terms in a partial fraction expansion are easier to integrate, the coefficients of those terms are easy to compute.

Let's try this approach with the generating function for Fibonacci numbers. First, we factor the denominator:

$$1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x)$$

where $\alpha_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\alpha_2 = \frac{1}{2}(1 - \sqrt{5})$. Next, we find A_1 and A_2 which satisfy:

$$\frac{x}{1 - x - x^2} = \frac{A_1}{1 - \alpha_1 x} + \frac{A_2}{1 - \alpha_2 x}$$

We do this by plugging in various values of x to generate linear equations in A_1 and A_2. We can then find A_1 and A_2 by solving a linear system. This gives:

$$A_1 = \frac{1}{\alpha_1 - \alpha_2} = \frac{1}{\sqrt{5}}$$

$$A_2 = \frac{-1}{\alpha_1 - \alpha_2} = -\frac{1}{\sqrt{5}}$$

Substituting into the equation above gives the partial fractions expansion of $F(x)$:

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha_1 x} - \frac{1}{1-\alpha_2 x}\right)$$

Each term in the partial fractions expansion has a simple power series given by the geometric sum formula:

$$\begin{aligned}\frac{1}{1-\alpha_1 x} &= 1 + \alpha_1 x + \alpha_1^2 x^2 + \cdots \\ \frac{1}{1-\alpha_2 x} &= 1 + \alpha_2 x + \alpha_2^2 x^2 + \cdots\end{aligned}$$

Substituting in these series gives a power series for the generating function:

$$\begin{aligned}F(x) &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha_1 x} - \frac{1}{1-\alpha_2 x}\right) \\ &= \frac{1}{\sqrt{5}}\left((1 + \alpha_1 x + \alpha_1^2 x^2 + \cdots) - (1 + \alpha_2 x + \alpha_2^2 x^2 + \cdots)\right),\end{aligned}$$

so

$$\begin{aligned}f_n &= \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)\end{aligned}$$

This formula may be scary and astonishing —it's not even obvious that its value is an integer —but it's very useful. For example, it provides (via the repeated squaring method) a much more efficient way to compute Fibonacci numbers than crunching through the recurrence, and it also clearly reveals the exponential growth of these numbers.

17.2.3 Problems

Class Problems

Problem 17.1.
The famous mathematician, Fibonacci, has decided to start a rabbit farm to fill up his time while he's not making new sequences to torment future college students. Fibonacci starts his farm on month zero (being a mathematician), and at the start of month one he receives his first pair of rabbits. Each pair of rabbits takes a month to mature, and after that breeds to produce one new pair of rabbits each month. Fibonacci decides that in order never to run out of rabbits or money, every time a batch of new rabbits is born, he'll sell a number of newborn pairs equal to the total number of pairs he had three months earlier. Fibonacci is convinced that this way he'll never run out of stock.

(a) Define the number, r_n, of pairs of rabbits Fibonacci has in month n, using a recurrence relation. That is, define r_n in terms of various r_i where $i < n$.

(b) Let $R(x)$ be the generating function for rabbit pairs,

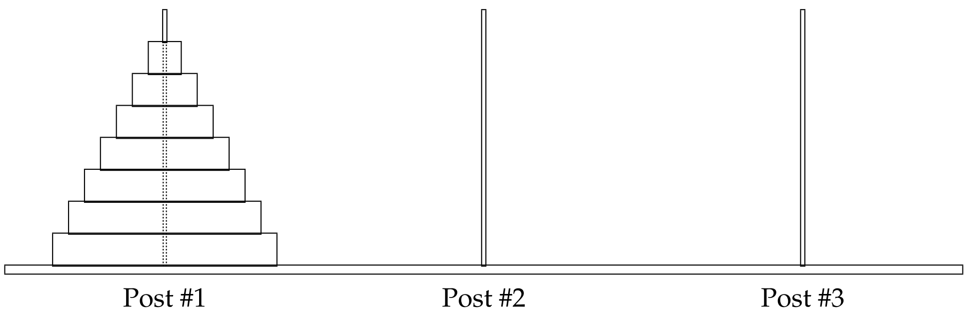
$$R(x) ::= r_0 + r_1 x + r_2 x^2 + \cdot.$$

Express $R(x)$ as a quotient of polynomials.

(c) Find a partial fraction decomposition of the generating function $R(x)$.

(d) Finally, use the partial fraction decomposition to come up with a closed form expression for the number of pairs of rabbits Fibonacci has on his farm on month n.

Problem 17.2.
Less well-known than the Towers of Hanoi —but no less fascinating —are the Towers of Sheboygan. As in Hanoi, the puzzle in Sheboygan involves 3 posts and n disks of different sizes. Initially, all the disks are on post #1:



The objective is to transfer all n disks to post #2 via a sequence of moves. A move consists of removing the top disk from one post and dropping it onto another post with the restriction that a larger disk can never lie above a smaller disk. Furthermore, a local ordinance requires that *a disk can be moved only from a post to the next post on its right —or from post #3 to post #1.* Thus, for example, moving a disk directly from post #1 to post #3 is not permitted.

(a) One procedure that solves the Sheboygan puzzle is defined recursively: to move an initial stack of n disks to the next post, move the top stack of $n - 1$ disks to the furthest post by moving it to the next post two times, then move the big, nth disk to the next post, and finally move the top stack another two times to land on top of the big disk. Let s_n be the number of moves that this procedure uses. Write a simple linear recurrence for s_n.

(b) Let $S(x)$ be the generating function for the sequence $\langle s_0, s_1, s_2, \ldots \rangle$. Show that $S(x)$ is a quotient of polynomials.

(c) Give a simple formula for s_n.

(d) A better (indeed optimal, but we won't prove this) procedure to solve the Towers of Sheboygan puzzle can be defined in terms of two mutually recursive procedures, procedure $P_1(n)$ for moving a stack of n disks 1 pole forward, and $P_2(n)$ for moving a stack of n disks 2 poles forward. This is trivial for $n = 0$. For $n > 0$, define:

$P_1(n)$: Apply $P_2(n-1)$ to move the top $n-1$ disks two poles forward to the third pole. Then move the remaining big disk once to land on the second pole. Then apply $P_2(n-1)$ again to move the stack of $n-1$ disks two poles forward from the third pole to land on top of the big disk.

$P_2(n)$: Apply $P_2(n-1)$ to move the top $n-1$ disks two poles forward to land on the third pole. Then move the remaining big disk to the second pole. Then apply $P_1(n-1)$ to move the stack of $n-1$ disks one pole forward to land on the first pole. Now move the big disk 1 pole forward again to land on the third pole. Finally, apply $P_2(n-1)$ again to move the stack of $n-1$ disks two poles forward to land on the big disk.

Let t_n be the number of moves needed to solve the Sheboygan puzzle using procedure $P_1(n)$. Show that

$$t_n = 2t_{n-1} + 2t_{n-2} + 3, \tag{17.4}$$

for $n > 1$.

Hint: Let s_n be the number of moves used by procedure $P_2(n)$. Express each of t_n and s_n as linear combinations of t_{n-1} and s_{n-1} and solve for t_n.

(e) Derive values a, b, c, α, β such that

$$t_n = a\alpha^n + b\beta^n + c.$$

Conclude that $t_n = o(s_n)$.

Homework Problems

Problem 17.3.
Taking derivatives of generating functions is another useful operation. This is done termwise, that is, if

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots ,$$

then

$$F'(x) ::= f_1 + 2f_2 x + 3f_3 x^2 + \cdots .$$

For example,

$$\frac{1}{(1-x)^2} = \left(\frac{1}{(1-x)}\right)' = 1 + 2x + 3x^2 + \cdots$$

so

$$H(x) ::= \frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + \cdots$$

is the generating function for the sequence of nonegative integers. Therefore

$$\frac{1+x}{(1-x)^3} = H'(x) = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots,$$

so

$$\frac{x^2+x}{(1-x)^3} = xH'(x) = 0 + 1x + 2^2x^2 + 3^2x^3 + \cdots + n^2x^n + \cdots$$

is the generating function for the nonegative integer squares.

(a) Prove that for all $k \in \mathbb{N}$, the generating function for the nonnegative integer kth powers is a quotient of polynomials in x. That is, for all $k \in \mathbb{N}$ there are polynomials $R_k(x)$ and $S_k(x)$ such that

$$[x^n]\left(\frac{R_k(x)}{S_k(x)}\right) = n^k. \tag{17.5}$$

Hint: Observe that the derivative of a quotient of polynomials is also a quotient of polynomials. It is not necessary work out explicit formulas for R_k and S_k to prove this part.

(b) Conclude that if $f(n)$ is a function on the nonnegative integers defined recursively in the form

$$f(n) = af(n-1) + bf(n-2) + cf(n-3) + p(n)\alpha^n$$

where the $a, b, c, \alpha \in \mathbb{C}$ and p is a polynomial with complex coefficients, then the generating function for the sequence $f(0), f(1), f(2), \ldots$ will be a quotient of polynomials in x, and hence there is a closed form expression for $f(n)$.

Hint: Consider

$$\frac{R_k(\alpha x)}{S_k(\alpha x)}$$

Problem 17.4.

Generating functions provide an interesting way to count the number of strings of matched parentheses. To do this, we'll use the description of these strings given in Definition 11.1.2 as the set, GoodCount, of strings of parentheses with a good count. Let c_n be the number of strings in GoodCount with exactly n left parentheses, and let $C(x)$ be the generating function for these numbers:

$$C(x) ::= c_0 + c_1x + c_2x^2 + \cdots.$$

(a) The *wrap* of a string, s, is the string, (s), that starts with a left parenthesis followed by the characters of s, and then ends with a right parenthesis. Explain why the generating function for the wraps of strings with a good count is $xC(x)$.

Hint: The wrap of a string with good count also has a good count that starts and ends with 0 and remains *positive* everywhere else.

(b) Explain why, for every string, s, with a good count, there is a unique sequence of strings $s_1, \ldots, s_k$ that are wraps of strings with good counts and $s = s_1 \cdots s_k$. For example, the string $r ::= (())()(()()) \in \text{GoodCount}$ equals $s_1 s_2 s_3$ where $s_1 = (()), s_2 = (), s_3 = (()())$, and this is the only way to express r as a sequence of wraps of strings with good counts.

(c) Conclude that

$$C = 1 + xC + (xC)^2 + \cdots + (xC)^n + \cdots , \tag{17.6}$$

so

$$C = \frac{1}{1 - xC} , \tag{17.7}$$

and hence

$$C = \frac{1 \pm \sqrt{1 - 4x}}{2x} . \tag{17.8}$$

Let $D(x) ::= 2xC(x)$. Expressing D as a power series

$$D(x) = d_0 + d_1 x + d_2 x^2 + \cdots ,$$

we have

$$c_n = \frac{d_{n+1}}{2}. \tag{17.9}$$

(d) Use (17.12), (17.13), and the value of c_0 to conclude that

$$D(x) = 1 - \sqrt{1 - 4x}.$$

(e) Prove that

$$d_n = \frac{(2n-3) \cdot (2n-5) \cdots 5 \cdot 3 \cdot 1 \cdot 2^n}{n!}.$$

Hint: $d_n = D^{(n)}(0)/n!$

(f) Conclude that

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Exam Problems

Problem 17.5.
Define the sequence $r_0, r_1, r_2, \ldots$ recursively by the rule that $r_0 = r_1 = 0$ and

$$r_n = 7r_{n-1} + 4r_{n-2} + (n+1),$$

for $n \geq 2$. Express the generating function of this sequence as a quotient of polynomials or products of polynomials. You do *not* have to find a closed form for r_n.

17.3 Counting with Generating Functions

Generating functions are particularly useful for solving counting problems. In particular, problems involving choosing items from a set often lead to nice generating functions by letting the coefficient of x^n be the number of ways to choose n items.

17.3.1 Choosing Distinct Items from a Set

The generating function for binomial coefficients follows directly from the Binomial Theorem:

$$\begin{aligned}\left\langle \binom{k}{0}, \binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{k}, 0, 0, 0, \ldots \right\rangle \quad &\longleftrightarrow \quad \binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 + \cdots + \binom{k}{k}x^k \\ &= \quad (1+x)^k\end{aligned}$$

Thus, the coefficient of x^n in $(1+x)^k$ is $\binom{k}{n}$, the number of ways to choose n distinct items from a set of size k. For example, the coefficient of x^2 is $\binom{k}{2}$, the number of ways to choose 2 items from a set with k elements. Similarly, the coefficient of x^{k+1} is the number of ways to choose $k+1$ items from a size k set, which is zero. (Watch out for this reversal of the roles that k and n played in earlier examples; we're led to this reversal because we've been using n to refer to the power of x in a power series.)

17.3.2 Building Generating Functions that Count

Often we can translate the description of a counting problem directly into a generating function for the solution. For example, we could figure out that $(1+x)^k$ generates the number of ways to select n distinct items from a k-element set without resorting to the Binomial Theorem or even fussing with binomial coefficients!

Here is how. First, consider a single-element set $\{a_1\}$. The generating function for the number of ways to select n elements from this set is simply $1+x$: we have 1 way to select zero elements, 1 way to select one element, and 0 ways to select more than one element. Similarly, the number of ways to select n elements from the set $\{a_2\}$ is also given by the generating function $1+x$. The fact that the elements differ in the two cases is irrelevant.

Now here is the main trick: *the generating function for choosing elements from a union of disjoint sets is the product of the generating functions for choosing from each set.* We'll justify this in a moment, but let's first look at an example. According to this principle, the generating function for the number of ways to select n elements from the $\{a_1, a_2\}$ is:

$$\underbrace{(1+x)}_{\substack{\text{gen func for}\\ \text{selecting an } a_1}} \cdot \underbrace{(1+x)}_{\substack{\text{gen func for}\\ \text{selecting an } a_2}} = \underbrace{(1+x)^2}_{\substack{\text{gen func for}\\ \text{selecting from}\\ \{a_1, a_2\}}} = 1 + 2x + x^2$$

Sure enough, for the set $\{a_1, a_2\}$, we have 1 way to select zero elements, 2 ways to select one element, 1 way to select two elements, and 0 ways to select more than two elements.

Repeated application of this rule gives the generating function for selecting n items from a k-element set $\{a_1, a_2, \ldots, a_k\}$:

$$\underbrace{(1+x)}_{\substack{\text{gen func for}\\ \text{selecting an } a_1}} \cdot \underbrace{(1+x)}_{\substack{\text{gen func for}\\ \text{selecting an } a_2}} \cdots \underbrace{(1+x)}_{\substack{\text{gen func for}\\ \text{selecting an } a_k}} = \underbrace{(1+x)^k}_{\substack{\text{gen func for}\\ \text{selecting from}\\ \{a_1, a_2, \ldots, a_k\}}}$$

This is the same generating function that we obtained by using the Binomial Theorem. But this time around we translated directly from the counting problem to the generating function.

We can extend these ideas to a general principle:

Rule 16 (Convolution Rule)**.** *Let $A(x)$ be the generating function for selecting items from set $\mathcal{A}$, and let $B(x)$ be the generating function for selecting items from set $\mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are disjoint, then the generating function for selecting items from the union $\mathcal{A} \cup \mathcal{B}$ is the product $A(x) \cdot B(x)$.*

This rule is rather ambiguous: what exactly are the rules governing the selection of items from a set? Remarkably, the Convolution Rule remains valid under *many* interpretations of selection. For example, we could insist that distinct items be selected or we might allow the same item to be picked a limited number of times or any number of times. Informally, the only restrictions are that (1) the order in which items are selected is disregarded and (2) restrictions on the selection of items from sets $\mathcal{A}$ and $\mathcal{B}$ also apply in selecting items from $\mathcal{A} \cup \mathcal{B}$. (Formally, there must be a bijection between n-element selections from $\mathcal{A} \cup \mathcal{B}$ and ordered pairs of selections from $\mathcal{A}$ and $\mathcal{B}$ containing a total of n elements.)

To count the number of ways to select n items from $\mathcal{A} \cup \mathcal{B}$, we observe that we can select n items by choosing j items from $\mathcal{A}$ and $n - j$ items from $\mathcal{B}$, where j is any number from 0 to n. This can be done in $a_j b_{n-j}$ ways. Summing over all the possible values of j gives a total of

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

ways to select n items from $\mathcal{A} \cup \mathcal{B}$. By the Product Rule, this is precisely the coefficient of x^n in the series for $A(x)B(x)$.

17.3.3 Choosing Items with Repetition

The first counting problem we considered was the number of ways to select a dozen doughnuts when five flavors were available. We can generalize this question as follows: in how many ways can we select n items from a k-element set if we're allowed to pick the same item multiple times? In these terms, the doughnut problem asks in how many ways we can select $n = 12$ doughnuts from the set of $k = 5$ flavors

$$\{\text{chocolate}, \text{lemon-filled}, \text{sugar}, \text{glazed}, \text{plain}\}$$

where, of course, we're allowed to pick several doughnuts of the same flavor. Let's approach this question from a generating functions perspective.

Suppose we make n choices (with repetition allowed) of items from a set containing a single item. Then there is one way to choose zero items, one way to choose one item, one way to choose two items, etc. Thus, the generating function for choosing n elements with repetition from a 1-element set is:

$$\begin{aligned} \langle 1, 1, 1, 1, \dots \rangle \quad &\longleftrightarrow \quad 1 + x + x^2 + x^3 + \cdots \\ &= \quad \frac{1}{1-x} \end{aligned}$$

The Convolution Rule says that the generating function for selecting items from a union of disjoint sets is the product of the generating functions for selecting items from each set:

$$\underbrace{\frac{1}{1-x}}_{\substack{\text{gen func for} \\ \text{choosing } a_1\text{'s}}} \cdot \underbrace{\frac{1}{1-x}}_{\substack{\text{gen func for} \\ \text{choosing } a_2\text{'s}}} \cdots \underbrace{\frac{1}{1-x}}_{\substack{\text{gen func for} \\ \text{choosing } a_k\text{'s}}} = \underbrace{\frac{1}{(1-x)^k}}_{\substack{\text{gen func for} \\ \text{repeated choice from} \\ \{a_1, a_2, \dots, a_k\}}}$$

Therefore, the generating function for choosing items from a k-element set with repetition allowed is $1/(1-x)^k$.

Now the Bookkeeper Rule tells us that the number of ways to choose n items with repetition from an k element set is

$$\binom{n+k-1}{n},$$

so this is the coefficient of x^n in the series expansion of $1/(1-x)^k$.

On the other hand, it's instructive to derive this coefficient algebraically, which we can do using Taylor's Theorem:

Theorem 17.3.1 (Taylor's Theorem)**.**

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots .$$

This theorem says that the nth coefficient of $1/(1-x)^k$ is equal to its nth derivative evaluated at 0 and divided by $n!$. Computing the nth derivative turns out not to be very difficult (Problem 17.7).

17.3.4 Problems

Practice Problems

Problem 17.6.
You would like to buy a bouquet of flowers. You find an online service that will make bouquets of **lilies**, **roses** and **tulips**, subject to the following constraints:

- there must be at most 3 lilies,
- there must be an odd number of tulips,
- there can be any number of roses.

Example: A bouquet of 3 tulips, 5 roses and no lilies satisfies the constraints.
Let f_n be the number of possible bouquets with n flowers that fit the service's constraints. Express $F(x)$, the generating function corresponding to $\langle f_0, f_1, f_2, \ldots \rangle$, as a quotient of polynomials (or products of polynomials). You do not need to simplify this expression.

Class Problems

Problem 17.7.
Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then it's easy to check that

$$a_n = \frac{A^{(n)}(0)}{n!},$$

where $A^{(n)}$ is the nth derivative of A. Use this fact (which you may assume) instead of the Convolution Counting Principle, to prove that

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n.$$

So if we didn't already know the Bookkeeper Rule, we could have proved it from this calculation and the Convolution Rule for generating functions.

Problem 17.8.
We are interested in generating functions for the number of different ways to compose a bag of n donuts subject to various restrictions. For each of the restrictions in (a)-(e) below, find a closed form for the corresponding generating function.

(a) All the donuts are chocolate and there are at least 3.

(b) All the donuts are glazed and there are at most 2.

(c) All the donuts are coconut and there are exactly 2 or there are none.

(d) All the donuts are plain and their number is a multiple of 4.

(e) The donuts must be chocolate, glazed, coconut, or plain and:

- there must be at least 3 chocolate donuts, and
- there must be at most 2 glazed, and
- there must be exactly 0 or 2 coconut, and
- there must be a multiple of 4 plain.

(f) Find a closed form for the number of ways to select n donuts subject to the constraints of the previous part.

Problem 17.9. **(a)** Let

$$S(x) ::= \frac{x^2 + x}{(1-x)^3}.$$

What is the coefficient of x^n in the generating function series for $S(x)$?

(b) Explain why $S(x)/(1-x)$ is the generating function for the sums of squares. That is, the coefficient of x^n in the series for $S(x)/(1-x)$ is $\sum_{k=1}^{n} k^2$.

(c) Use the previous parts to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Homework Problems

Problem 17.10.
We will use generating functions to determine how many ways there are to use pennies, nickels, dimes, quarters, and half-dollars to give n cents change.

(a) Write the sequence P_n for the number of ways to use only pennies to change n cents. Write the generating function for that sequence.

(b) Write the sequence N_n for the number of ways to use only nickels to change n cents. Write the generating function for that sequence.

(c) Write the generating function for the number of ways to use only nickels and pennies to change n cents.

(d) Write the generating function for the number of ways to use pennies, nickels, dimes, quarters, and half-dollars to give n cents change.

(e) Explain how to use this function to find out how many ways are there to change 50 cents; you do *not* have to provide the answer or actually carry out the process.

Exam Problems

Problem 17.11.
The working days in the next year can be numbered 1, 2, 3, ..., 300. I'd like to avoid as many as possible.

- On even-numbered days, I'll say I'm sick.
- On days that are a multiple of 3, I'll say I was stuck in traffic.
- On days that are a multiple of 5, I'll refuse to come out from under the blankets.

In total, how many work days will I *avoid* in the coming year?

Problem 17.12.
Define the sequence $r_0, r_1, r_2, \ldots$ recursively by the rule that $r_0 = r_1 = 0$ and

$$r_n = 7r_{n-1} + 4r_{n-2} + (n+1),$$

for $n \geq 2$. Express the generating function of this sequence as a quotient of polynomials or products of polynomials. You do *not* have to find a closed form for r_n.

Problem 17.13.
Find the coefficients of $x^{10}y^5$ in $(19x + 4y)^{15}$

17.4 An "Impossible" Counting Problem

So far everything we've done with generating functions we could have done another way. But here is an absurd counting problem— really over the top! In how many ways can we fill a bag with n fruits subject to the following constraints?

- The number of apples must be even.

- The number of bananas must be a multiple of 5.
- There can be at most four oranges.
- There can be at most one pear.

For example, there are 7 ways to form a bag with 6 fruits:

Apples	6	4	4	2	2	0	0
Bananas	0	0	0	0	0	5	5
Oranges	0	2	1	4	3	1	0
Pears	0	0	1	0	1	0	1

These constraints are so complicated that the problem seems hopeless! But let's see what generating functions reveal.

Let's first construct a generating function for choosing apples. We can choose a set of 0 apples in one way, a set of 1 apple in zero ways (since the number of apples must be even), a set of 2 apples in one way, a set of 3 apples in zero ways, and so forth. So we have:

$$A(x) = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1 - x^2}$$

Similarly, the generating function for choosing bananas is:

$$B(x) = 1 + x^5 + x^{10} + x^{15} + \cdots = \frac{1}{1 - x^5}$$

Now, we can choose a set of 0 oranges in one way, a set of 1 orange in one way, and so on. However, we can not choose more than four oranges, so we have the generating function:

$$O(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1 - x^5}{1 - x}$$

Here we're using the geometric sum formula. Finally, we can choose only zero or one pear, so we have:

$$P(x) = 1 + x$$

The Convolution Rule says that the generating function for choosing from among all four kinds of fruit is:

$$\begin{aligned} A(x)B(x)O(x)P(x) &= \frac{1}{1 - x^2}\,\frac{1}{1 - x^5}\,\frac{1 - x^5}{1 - x}(1 + x) \\ &= \frac{1}{(1 - x)^2} \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots \end{aligned}$$

Almost everything cancels! We're left with $1/(1 - x)^2$, which we found a power series for earlier: the coefficient of x^n is simply $n + 1$. Thus, the number of ways to form a bag of n fruits is just $n + 1$. This is consistent with the example we worked out, since there were 7 different fruit bags containing 6 fruits. *Amazing!*

17.4.1 Problems

Homework Problems

Problem 17.14.
Miss McGillicuddy never goes outside without a collection of pets. In particular:

- She brings a positive number of songbirds, which always come in pairs.
- She may or may not bring her alligator, Freddy.
- She brings at least 2 cats.
- She brings two or more chihuahuas and labradors leashed together in a line.

Let P_n denote the number of different collections of n pets that can accompany her, where we regard chihuahuas and labradors leashed up in different orders as different collections, even if there are the same number chihuahuas and labradors leashed in the line.

For example, $P_6 = 4$ since there are 4 possible collections of 6 pets:

- 2 songbirds, 2 cats, 2 chihuahuas leashed in line
- 2 songbirds, 2 cats, 2 labradors leashed in line
- 2 songbirds, 2 cats, a labrador leashed behind a chihuahua
- 2 songbirds, 2 cats, a chihuahua leashed behind a labrador

And $P_7 = 16$ since there are 16 possible collections of 7 pets:

- 2 songbirds, 3 cats, 2 chihuahuas leashed in line
- 2 songbirds, 3 cats, 2 labradors leashed in line
- 2 songbirds, 3 cats, a labrador leashed behind a chihuahua
- 2 songbirds, 3 cats, a chihuahua leashed behind a labrador
- 4 collections consisting of 2 songbirds, 2 cats, 1 alligator, and a line of 2 dogs
- 8 collections consisting of 2 songbirds, 2 cats, and a line of 3 dogs.

(a) Let
$$P(x) ::= P_0 + P_1 x + P_2 x^2 + P_3 x^3 + \cdots$$
be the generating function for the number of Miss McGillicuddy's pet collections. Verify that
$$P(x) = \frac{4x^6}{(1-x)^2(1-2x)}.$$

(b) Find a simple formula for P_n.

Problem 17.15.
Generating functions provide an interesting way to count the number of strings of matched parentheses. To do this, we'll use the description of these strings given in Definition 11.1.2 as the set, GoodCount, of strings of parentheses with a good count. Let c_n be the number of strings in GoodCount with exactly n left parentheses, and let $C(x)$ be the generating function for these numbers:

$$C(x) ::= c_0 + c_1 x + c_2 x^2 + \cdots .$$

(a) The *wrap* of a string, s, is the string, (s), that starts with a left parenthesis followed by the characters of s, and then ends with a right parenthesis. Explain why the generating function for the wraps of strings with a good count is $xC(x)$.

Hint: The wrap of a string with good count also has a good count that starts and ends with 0 and remains *positive* everywhere else.

(b) Explain why, for every string, s, with a good count, there is a unique sequence of strings $s_1, \ldots, s_k$ that are wraps of strings with good counts and $s = s_1 \cdots s_k$. For example, the string $r ::= (())()(()()) \in$ GoodCount equals $s_1 s_2 s_3$ where $s_1 = (())$, $s_2 = ()$, $s_3 = (()())$, and this is the only way to express r as a sequence of wraps of strings with good counts.

(c) Conclude that

$$C = 1 + xC + (xC)^2 + \cdots + (xC)^n + \cdots , \tag{17.10}$$

so

$$C = \frac{1}{1 - xC} , \tag{17.11}$$

and hence

$$C = \frac{1 \pm \sqrt{1 - 4x}}{2x} . \tag{17.12}$$

Let $D(x) ::= 2xC(x)$. Expressing D as a power series

$$D(x) = d_0 + d_1 x + d_2 x^2 + \cdots ,$$

we have

$$c_n = \frac{d_{n+1}}{2}. \tag{17.13}$$

(d) Use (17.12), (17.13), and the value of c_0 to conclude that

$$D(x) = 1 - \sqrt{1 - 4x}.$$

(e) Prove that

$$d_n = \frac{(2n-3) \cdot (2n-5) \cdots 5 \cdot 3 \cdot 1 \cdot 2^n}{n!}.$$

Hint: $d_n = D^{(n)}(0)/n!$

(f) Conclude that

$$c_n = \frac{1}{n+1} \binom{2n}{n}.$$

Exam Problems

Problem 17.16.
T-Pain is planning an epic boat trip and he needs to decide what to bring with him.

- He definitely wants to bring burgers, but they only come in packs of 6.
- He and his two friends can't decide whether they want to dress formally or casually. He'll either bring 0 pairs of flip flops or 3 pairs.
- He doesn't have very much room in his suitcase for towels, so he can bring at most 2.
- In order for the boat trip to be truly epic, he has to bring at least 1 nautical-themed pashmina afghan.

(a) Let g_n be the the number of different ways for T-Pain to bring n items (burgers, pairs of flip flops, towels, and/or afghans) on his boat trip. Express the generating function $G(x) ::= \sum_{n=0}^{\infty} g_n x^n$ as a quotient of polynomials.

(b) Find a closed formula in n for the number of ways T-Pain can bring exactly n items with him.