Test Code: PCB (short answer type) 2015

M.Tech. in Computer Science

Syllabus and Sample Questions

The selection test for M.Tech. in Computer Science will consist of two parts.

- Test **MMA** (objective type) in the forenoon session is the 1^{st} part, and
- Test **PCB** (short answer type) in the afternoon session is the 2^{nd} part.

  The **PCB** test will consist of two groups.

  ◇ **Group A** (28 Marks) : All candidates have to answer questions on analytical ability and mathematics at the undergraduate level.

  ◇ **Group B** (72 Marks) : A candidate has *to choose exactly one of the following five sections*, from which questions have to be answered:
  (i) Mathematics, (ii) Statistics, (iii) Physics, (iv) Computer Science, and (v) Engineering and Technology.
  While questions in the first three sections will be at postgraduate level, those for the last two sections will be at B.Tech. level.

The syllabus and sample questions for the **MMA** test are available separately. The syllabus and sample questions for the **PCB** test are given below.

Note:

1. Not all questions in this sample set are of equal difficulty. They may not carry equal marks in the test. More sample questions are available on the website for M.Tech(CS) at *http://www.isical.ac.in/~deanweb/MTECHCSSQ.html*
2. Each of the two tests **MMA** and **PCB**, will have individual qualifying marks.

SYLLABUS for Test PCB

Group A

Short answer type test based on the syllabus of MMA.

Group B

Mathematics

Calculus and real analysis – real numbers, basic properties, convergence of sequences and series, limits, continuity, uniform continuity of functions, differentiability of functions of one or more variables and applications, indefinite integral, fundamental theorem of Calculus, Riemann integration, improper integrals, double and multiple integrals and applications, sequences and series of functions, uniform convergence.
Linear algebra – vector spaces and linear transformations, matrices and systems of linear equations, characteristic roots and characteristic vectors, Cayley-Hamilton theorem, canonical forms, quadratic forms.
Graph Theory – connectedness, trees, vertex coloring, planar graphs, Eulerian graphs, Hamiltonian graphs, digraphs and tournaments.
Abstract algebra – groups, subgroups, cosets, Lagrange's theorem, normal subgroups and quotient groups, permutation groups, rings, subrings, ideals, integral domains, fields, characteristics of a field, polynomial rings, unique factorization domains, field extensions, finite fields.
Differential equations – solutions of ordinary and partial differential equations and applications.

Statistics

Notions of sample space and probability, combinatorial probability, conditional probability, Bayes' theorem and independence.
Random variable and expectation, moments, standard univariate discrete and continuous distributions, sampling distribution of statistics based on normal samples, central limit theorem, approximation of binomial to normal, Poisson law.
Multinomial, bivariate normal and multivariate normal distributions.
Descriptive statistical measures, product-moment correlation, partial and multiple correlation.
Regression – simple and multiple.
Elementary theory and methods of estimation – unbiasedness, minimum variance, sufficiency, maximum likelihood method, method of moments, least squares methods.
Tests of hypotheses – basic concepts and simple applications of Neyman-Pearson lemma, confidence intervals.
Tests of regression, elements of non-parametric inference, contingency tables and Chi-square, ANOVA, basic designs (CRD/RBD/LSD) and their analyses, elements of factorial designs.
Conventional sampling techniques, ratio and regression methods of estimation.

Physics

General properties of matter – elasticity, surface tension, viscosity.
Classical dynamics – Lagrangian and Hamiltonian formulation, symmetries and conservation laws, motion in central field of force, planetary motion, collision and scattering, mechanics of system of particles, small oscillation and normal modes, wave motion, special theory of relativity.
Electrodynamics – electrostatics, magnetostatics, electromagnetic induction, self and mutual inductance, capacitance, Maxwell's equation in free space.
Nonrelativistic quantum mechanics – Planck's law, photoelectric effect, Compton effect, wave-particle duality, Heisenberg's uncertainty principle, Schrodinger's equation, and some applications.
Thermodynamics and statistical Physics – laws of thermodynamics and their consequences, thermodynamic potentials and Maxwell's relations, chemical potential, phase equilibrium, phase space, microstates and macrostates, partition function free energy, classical statistics.
Atomic and molecular physics – quantum states of an electron in an atom, Hydrogen atom spectrum, electron spin, spin-orbit coupling, fine structure, Zeeman effect.
Condensed matter physics – crystal classes, 2D and 3D lattice, reciprocal lattice, bonding, diffraction and structure factor, point defects and dislocations, lattice vibration, free electron theory, electron motion in periodic potential, energy bands in metals, insulators and semiconductors.
Nuclear and particle physics – Basics of nuclear properties, nuclear forces, nuclear structures, nuclear reactions, interaction of charged particles and e-m waves with matter, theoretical understanding of radioactive decay, particle physics at the elementary level.
Electronics – semiconductor physics; diodes - clipping, clamping, rectification; Zener regulated power supply, bipolar junction transistor - CC, CB, and CE configuration; transistor as a switch; amplifiers.
Operational Amplifier and its applications – inverting, noninverting amplifiers, adder, integrator, differentiator, waveform generator comparator, Schmidt trigger.
Digital integrated circuits – NAND, NOR gates as building blocks, XOR gates, combinational circuits, half and full adder.

Computer Science

Data structures - array, stack, queue, linked list, binary tree, heap, AVL tree, B-tree.
Discrete Mathematics - recurrence relations, generating functions, graph theory - paths and cycles, connected components, trees, digraphs.
Programming languages - Fundamental concepts - abstract data types, procedure call and parameter passing, languages like C and C++.

Design and analysis of algorithms - Asymptotic notation, sorting, selection, searching, graph traversal, minimum spanning tree.
Switching Theory and Logic Design - Boolean algebra, minimization of Boolean functions, combinational and sequential circuits - synthesis and design.
Computer organization and architecture - Number representation, computer arithmetic, memory organization, I/O organization, microprogramming, pipelining, instruction level parallelism.
Operating systems - Memory management, processor management, critical section problem, deadlocks, device management, file systems.
Formal languages and automata theory - Finite automata and regular expressions, pushdown automata, context-free grammars, Turing machines, elements of undecidability.
Database management systems - Relational model, relational algebra, relational calculus, functional dependency, normalization (up to 3-rd normal form).
Computer networks - OSI, LAN technology - Bus/tree, Ring, Star; MAC protocols; WAN technology - circuit switching, packet switching; data communications - data encoding, routing, flow control, error detection/correction, Internet working, TCP/IP networking including IPv4.

Engineering and Technology

C Programming language.
Gravitation, moments of inertia, particle dynamics, elasticity, friction, strength of materials, surface tension and viscosity.
Laws of thermodynamics and heat engines.
Electrostatics, magnetostatics and electromagnetic induction.
Laws of electrical circuits – transient and steady state responses of resistive and reactive circuits.
D.C. generators, D.C. motors, induction motors, alternators, transformers.
Diode circuits, bipolar & FET devices and circuits, transistor circuits, oscillator, multi-vibrator, operational amplifier.
Digital circuits – combinatorial and sequential circuits, multiplexer, de-multiplexer, counter, A/D and D/A converters.
Boolean algebra, minimization of switching functions.

SAMPLE QUESTIONS

Group A

A1. How many times will the digit '7' be written when listing the integers from 1 to 1000? Justify your answer.

A2. For sets A and B, define $A\Delta B = (\bar{A} \cap B) \cup (A \cap \bar{B})$. Show the following for any three sets A, B and C.

(a) $A\Delta A = \emptyset$.

(b) $A\Delta(B\Delta C) = (A\Delta B)\Delta C$.

(c) If $A\Delta B = A\Delta C$ then $B = C$.

A3. In a group of n persons, each person is asked to write down the sum of the ages of all the other $(n-1)$ persons. Suppose the sums so obtained are $s_1, \ldots, s_n$. It is now desired to find the actual ages of the persons from these values.

(a) Formulate the problem in the form of a system of linear equations.

(b) Can the ages be always uniquely determined? Justify your answer.

A4. Evaluate

$$\lim_{x \to 0} \left(x^2 \left(1 + 2 + 3 + \cdots + \left[\frac{1}{|x|} \right] \right) \right) .$$

For any real number a, $[a]$ is the largest integer not greater than a.

A5. For $n \geq 4$, prove that $1! + 2! + \cdots + n!$ cannot be the square of a positive integer.

A6. Let *a, b* and *c* be the three sides of a triangle. Show that

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3 .$$

A7. Find all pairs of prime numbers *p, q* such that $p + q = 18(p - q)$. Justify your answer.

A8. Suppose P and Q are $n \times n$ matrices of real numbers such that

- $P^2 = P$,
- $Q^2 = Q$, and
- $I - P - Q$ is invertible, where I is a $n \times n$ identity matrix.

Show that P and Q have the same rank.

Group B

Computer Science

C1. (a) How many asterisks ($\star$) in terms of k will be printed by the following C function, when called as $count(m)$ where $m = 3^k$? Justify your answer. Assume that 4 bytes are used to store an integer in C and k is such that 3^k can be stored in 4 bytes.

```
void count(int n)
{
     printf("*");
     if(n>1)
     {
          count(n/3);
          count(n/3);
          count(n/3);
     }
}
```

(b) A 64000-byte message is to be transmitted over a 2-hop path in a store-and-forward packet-switching network. The network limits packets to a maximum size of 2032 bytes including a 32-byte header. The transmission lines in the network are error free and have a speed of 50 Mbps. Each hop is 1000 km long and the signal propagates at the speed of light (3×10^8 meters per second). Assume that queuing and processing delays at the intermediate node are negligible. How long does it take to deliver the entire message from the source to the destination?

C2. Give an efficient implementation for a data structure `STACK_MAX` to support an operation `max` that reports the current maximum among all elements in the stack. Usual stack operations (`createEmpty`, `push`, `pop`) are also to be supported.

How many bytes are needed to store your data structure after the following operations: `createEmpty`, `push`(5), `push`(6), `push`(7), `pop`, `max`, `push`(6), `push`(8), `pop`, `pop`, `max`, `push`(5). Assume that an integer can be stored in 4 bytes.

C3. You are given an array $X[\,]$. The size of the array is very large but unknown. The first few elements of the array are distinct positive integers in sorted order. The rest of the elements are 0. The number of positive integers in the array is also not known.

Design an algorithm that takes a positive integer y as input and finds the position of y in X. Your algorithm should return "Not found" if y is not in the array. You will get no credit if the complexity of your algorithm is linear (or higher) in the number of positive integers in X.

C4. (a) Prove or disprove the following statement: *The union of a regular language with a disjoint non-regular language over the same alphabet can never be regular.*

[Hint: You may use the closure properties of regular languages.]

(b) It is known that the language $L_1 = \{0^n 1^n 2^i \mid i \neq n\}$ is not a context free language (CFL). Now consider the language $L_2 = \{0^i 1^n 2^n \mid i \neq n\}$. We can prove L_2 is not a CFL by converting L_2 into L_1 by applying two operations, both known to be closed on CFLs. What are the two operations you will use for this conversion? Justify your answer.

C5. Consider three relations $R1(\underline{X}, Y, Z)$, $R2(\underline{M}, N, P)$, and $R3(\underline{N, X})$. The primary keys of the relations are underlined. The relations have 100, 30, and 400 tuples, respectively. The space requirements for different attributes are: $X = 30$ bytes, $Y = 10$ bytes, $Z = 10$ bytes, $M = 20$ bytes, $N = 20$ bytes, and $P = 10$ bytes. Let $V(A, R)$ signify the variety of values that attribute A may have in the relation R. Let $V(N, R2) = 15$ and $V(N, R3) = 300$. Assume that the distribution of values is uniform.

(a) If $R1$, $R2$, and $R3$ are to be joined, find the order of join for the minimum cost. The cost of a join is defined as the total space required by the intermediate relations. Justify your answer.

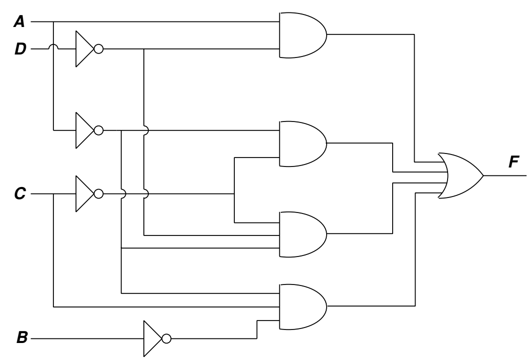
(b) Calculate the minimum number of disk accesses (including both reading the relations and writing the results) required to join $R1$ and $R3$ using block-oriented loop algorithm. Assume that (i) 10 tuples occupy a block and (ii) the smaller of the two relations can be totally accommodated in main memory during execution of the join.

C6. (a) Consider three processes, P_1, P_2, and P_3. Their start times and execution times are given below.

Process	Start time	Execution time
P_1	$t = 0$ ms	100 ms
P_2	$t = 25$ ms	50 ms
P_3	$t = 50$ ms	20 ms

Let Δ be the amount of time taken by the kernel to complete a context switch from any process P_i to P_j. For what values of Δ will the average turnaround time for P_1, P_2, P_3 be reduced by choosing a Shortest Remaining Time First scheduling policy over a Shortest Job First policy?

(b) The circuit shown in the following figure computes a Boolean function F. Assuming that all gates cost Rs. 5 per input (i.e., an inverter costs Rs. 5, a 2-input gate costs Rs. 10, etc.), find the minimum cost realization of F using only inverters, AND / OR gates.



C7. (a) Identifiers in a certain language have the following properties:

- they start with a lower case letter,
- they may contain upper case letters, but each uppercase letter must be followed by one or more lower case letters,
- they may contain digits but only at the end.

Thus, `num` and `varName1` are valid identifiers, but `aBC` and `a2i` are not. Write a regular expression for such identifiers. You may use extended notation if necessary.

(b) Consider the following grammar G.

$$\begin{array}{rcl} S & \rightarrow & L = E \\ E & \rightarrow & L \\ L & \rightarrow & \mathbf{id} \\ L & \rightarrow & \textit{Elist}\ ] \\ \textit{Elist} & \rightarrow & \mathbf{id}\ [\ E \\ \textit{Elist} & \rightarrow & \textit{Elist}, E \end{array}$$

S, L, E, and *Elist* are the non-terminals; all other symbols appearing in the above grammar are terminals. Construct an LL(1) grammar that is equivalent to G.

C8. (a) Let $a_{n-1}a_{n-2}\ldots a_0$ and $b_{n-1}b_{n-2}\ldots b_0$ denote the 2's complement representation of two integers A and B respectively. Addition of A and B yields a sum $S = s_{n-1}s_{n-2}\ldots s_0$. The outgoing carry generated at the most significant bit position, if any, is ignored. Show that an overflow (incorrect addition result) will occur only if the following Boolean condition holds:

$$\overline{s}_{n-1} \oplus (a_{n-1}s_{n-1}) = b_{n-1}(s_{n-1} \oplus a_{n-1})$$

where $\oplus$ denotes the Boolean XOR operation. You may use the Boolean identity: $X + Y = X \oplus Y \oplus (XY)$ to prove your result.

(b) Consider a machine with 5 stages F, D, X, M, W, where F denotes instruction fetch, D - instruction decode and register fetch, X - execute/address calculation, M - memory access, and W - write back to a register. The stage F needs 9 nanoseconds (ns), D needs 3 ns, X requires 7 ns, M needs 9 ns, and W takes 2 ns. Let M_1 denote a non-pipelined implementation of the machine, where each instruction has to be executed in a single clock cycle. Let M_2 denote a 5-stage pipelined version of the machine. Assume that pipeline overhead is 1 ns for each stage. Calculate the maximum clock frequency that can be used in M_1 and in M_2.

C9. (a) Read the C code given below. Use the four integers corresponding to the four digits of *your question booklet number* as input to the program. For example, if your question booklet number is 9830, then your input would be this: 9 8 3 0

What will the program print for your input?

```
#include<stdio.h>
#define STACKSIZE 2

typedef float Type;

typedef struct Ftype{
  int N;
  int D;
}F_inp;

typedef struct stack {
    F_inp item;
    int number;
}STACK;

STACK index[STACKSIZE];
STACK *ptr = index;

void PushF(int i, int j, int count){
```

```
        ptr[count].item.N = i;
        ptr[count].item.D = j;
        ptr[count].number = count+1;
    }

    Type Doit(int count){
        Type val;

        if(count == 0)  return(1.0);
        else{
         if ((Type)ptr[count-1].item.D == 0)
            return 1.0;
            val = (Type)ptr[count-1].item.N/
                  (Type)ptr[count-1].item.D;
            return(Doit(--count) * val);
        }
    }

    void main() {
        int i, j, count=0;

        while (count < STACKSIZE){
            scanf("%d%d",&i,&j);
            printf("%d%d\n", i, j);
            PushF(i,j,count);
            count++;
        }
    printf("The output is: %f, i.e., %3.2f\n",
                Doit(count), Doit(count));
    }
```

(b) You are given a 2-variable Boolean function $f(x_1, x_2)$ as follows:

$$f(x_1, x_2) = x_1 \oplus x_2 \oplus \overline{x_1}.\overline{x_2}$$

Express f in conjunctive normal form.

C10. (a) A *palindrome* over the alphabet $\Sigma = \{a, b, \ldots, z\}$, ($|\Sigma|$ = 26) is a string that reads the same both forwards and backwards. For example, **tenet** is a palindrome over Σ. Let $P(n)$ be the number of palindromes of length n over Σ. Derive an expression for $P(n)$ in terms of n. *You may use recurrence relations.*

(b) For any two languages $L_1, L_2 \subseteq \{0,1\}^*$, their symmetric difference SD(L_1, L_2) is the set of strings that are in exactly one of L_1 and L_2. For example, if L_1 = {00, 101} and L_2 = {11, 00}, then SD(L_1, L_2) = {11, 101}.

(i) Suppose A is the set of all strings of the form 0^*1^*, and B is the set of all strings of the form 1^*0^*.

- List all the strings of length 3 or less in SD (A, B).
- Write a regular expression for SD(A, B).

(ii) Is SD (L_1, L_2) necessarily a Context-Free language? Justify your answer.

C11. (a) You are given a sorted list A of n real numbers $a_1, a_2, \ldots, a_n$ with values in the range (α, β). Write an $O(n)$ time algorithm to partition A into two disjoint non-empty subsets A_1 and A_2 such that

$$\max_{a_i \in A_1} |\alpha - a_i| + \max_{a_j \in A_2} |\beta - a_j|$$

is minimum among all such possible partitions.

(b) Let $A[1 \ldots n]$ be a given array of n integers, where $n = 2^m$. The following two operations are the only ones to be applied to A:

- **Add**(i, y): Increment the value of $A[i]$ by y.
- **Partial-sum**(k): Print the current value of $\sum_{i=1}^{k} A[i]$.

One needs to perform these two operations multiple times in any given order. Design a data structure to store A such that each invocation of these two operations can be done in $O(m)$ steps.

C12. Consider a singly linked list, with each node containing an integer and a pointer to the next node. The last node of the list points to $NULL$. You are given two such lists A and B containing m and n nodes, respectively. An *intersection point* between two linked lists is a node common to both.

(i) Design an $O(m+n)$ algorithm to find whether there exists an intersection point between A and B.

(ii) If your algorithm in (i) above reports YES, then design an $O(m+n)$ algorithm to find the *first* intersection point between A and B.

You are not allowed to modify A and B. *Partial credit may be given if your algorithm uses more than* $\Theta(1)$ *additional space.*

C13. (a) Let R and S be two relations, and l be an attribute common to R and S. Let c be a condition over the attributes common to R and S. Prove or disprove the following:

(i) $\Pi_l(R - S) = \Pi_l(R) - \Pi_l(S)$;

(ii) $\sigma_c(R \bowtie S) = \sigma_c(R) \bowtie \sigma_c(S)$.

(b) Following are the steps executed by the CPU in a certain order, to process an interrupt received from a device. Mention the correct order of execution of these steps.

I. CPU executes the Interrupt Service Routine.

II. CPU uses the vector number to look up the address of the Interrupt Service Routine to be executed.

III. CPU returns to the point of execution where it was interrupted.

IV. Interrupt Service Routine restores the saved registers from the stack.

V. CPU grants the interrupt for the device and sends interrupt acknowledge to the device (IACK).

VI. Interrupt Service Routine saves the registers onto a stack.

VII. CPU receives the vector number from the device.

C14. (a) Consider two processes *P1* and *P2* entering the ready queue with the following properties:

- *P1* needs a total of 12 time units of CPU execution and 20 time units of I/O execution. After every 3 time units of CPU work, 5 time units of I/O are executed for *P1*.
- *P2* needs a total of 15 time units of CPU execution and no I/O. *P2* arrives just after *P1*.

Report the schedules, and the corresponding completion times of *P1* and *P2* for each of the following two scheduling strategies:

(i) Shortest Remaining Time First (preemptive), and

(ii) Round Robin, with a slice of 4 time units.

(b) What will happen to a packet sent to the IPv4 address 127.0.0.1?

(c) A 2km long LAN has 10Mbps bandwidth and uses CSMA/CD. The signal travels along the wire at 2×10^8m/s. What is the minimum packet size that can be used on this network?

C15. (a) Calculate how many integers in the set $\{1, 2, 3, ..., 1000\}$ are not divisible by 2, 5, or 11.

(b) Let 5 points be randomly placed in a square box of size 2×2. Show that the distance of the closest pair of these five points is at most $\sqrt{2}$.

(c) Show that, given $2^n + 1$ points with integer coordinates in $\mathbb{R}^n$, there exists a pair of points among them such that all the coordinates of the midpoint of the line segment joining them are integers.

(d) Find the number of functions from the set $\{1, 2, 3, 4, 5\}$ onto the set $\{1, 2, 3\}$.

Engineering and Technology

E1. (a) A 50kW compound generator works on half-load with a terminal voltage of 250V. The shunt, series and armature windings have resistances of 126Ω, 0.02Ω and 0.05Ω respectively. Calculate the total power generated at the armature when the machine is connected to short-shunt.

(b) A single phase 60kVA transformer delivers full load at 0.75 power factor with 90% efficiency. If the same transformer works at half load at 0.70 power factor, its efficiency increases to 91.3%. Calculate the iron loss of the transformer.

E2. Two long straight parallel wires stand 2 meters apart in air and carry currents I_1 and I_2 in the same direction. The field intensity at a point midway between the wires is 7.95 Ampere-turn per meter. The force on each wire per unit length is 2.4 $\times 10^{-4}$ N. Assume that the absolute permeability of air is $4\pi \times 10^{-7}$ H per meter.

(a) Explain the nature of the force experienced between the two wires, *i.e.* attractive or repulsive.

(b) Determine I_1 and I_2.

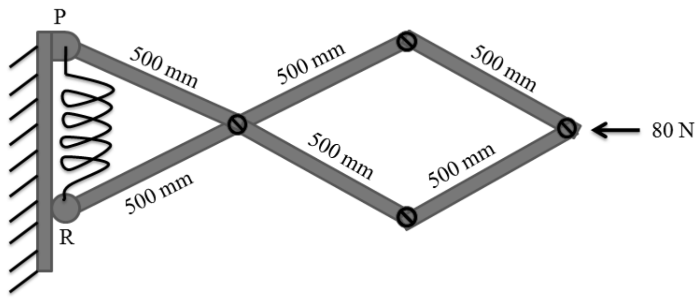
(c) Another parallel wire carrying a current of 50 A in the opposite direction is now placed midway between the two wires and in the same plane. Determine the resultant force on this wire.

E3. A choke coil connected across a 500 V, 50 Hz supply takes 1 A current at a power factor of 0.8.

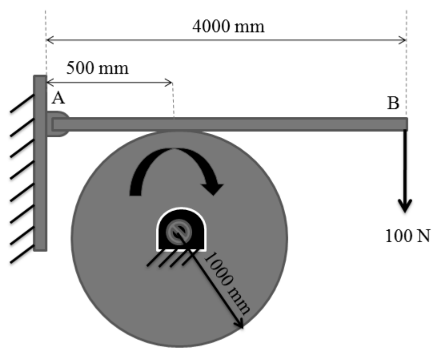
(a) Determine the capacitance that must be placed in series with the choke coil so that it resonates at 50 Hz.

(b) An additional capacitor is now connected in parallel with the above combination in (a) to change the resonant frequency. Obtain an expression for the additional capacitance in terms of the new resonant frequency.

E4. (a) The mechanical system shown in the figure below is loaded by a horizontal 80 N force. The length of the spring is 500 mm. Each arm of the mechanical system is also of length 500 mm as shown in the figure. Under the influence of 80 N load, the spring is stretched to 600 mm but the entire mechanical system including the spring remains in equilibrium. Determine the stiffness of the spring. Note that the spring and the frame are fixed at the pin position P. The other end of the spring is at R which is a frictionless roller free to move along the vertical axis. Assume that the mechanical joints between the arms are frictionless.



(b) A brake system is shown in the figure below. The solid disk of radius 1000 mm is being rotated at 196 rpm. The bar AB, of length 4000 mm, is fixed at the end A and subjected to a downward load of 100 N at the end B to stop the rotation of the disk. The bar AB (assumed to be horizontal) touches the rotating disk at a point 500 mm from the fixed end of the bar. The weight of the disk is 10 Kg and the coefficient of friction between the bar and the disk is 0.5. Calculate the number of revolutions the disk will make before coming to rest.



E5. (a) Air at 90°C and 605 Kg per square meter pressure is heated to 180°C keeping the volume constant at 21 cubic meter. Find

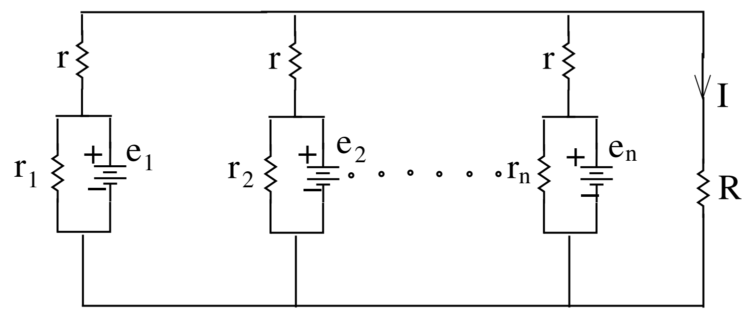
(i) the final pressure, and

(ii) the change in the internal energy.

Note that the specific heat at constant pressure (C_p), the specific heat at constant volume (C_v), and the mechanical equivalent of heat are 0.3, 0.2 and 420 Kg-meter per Kcal, respectively.

(b) A molten metal is forced through a cylindrical die at a pressure of 168×10^3 Kg per square meter. Given that the density of the molten metal is 2000 Kg per cubic meter and the specific heat of the metal is 0.03, find the rise in temperature during this process. Assume that the mechanical equivalent of heat is 420 Kg-meter per Kcal.

E6. (a) Calculate the current I flowing through the resistor R shown in the following figure ($e_1 < e_2 < \cdots < e_n$).

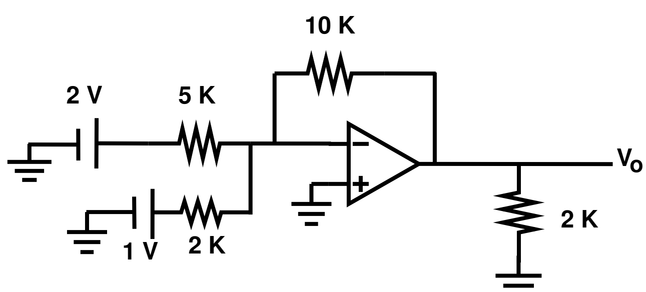


(b) A parallel plate capacitor is charged to 75 μC at 100 V. After removing the 100 V source, the capacitor is immediately connected to an uncharged capacitor with capacitance twice that of the first one. Determine the energy of the system before and after the connection is made. Assume that all capacitors are ideal.

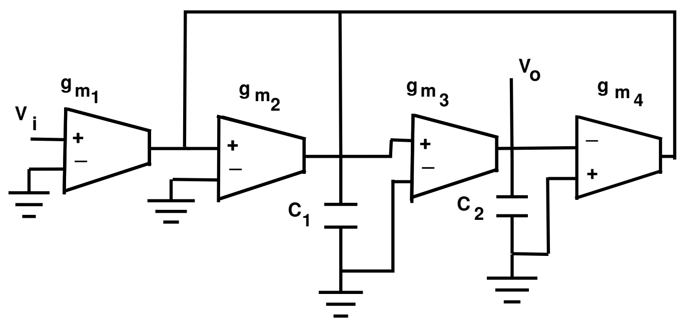
E7. (a) Draw Karnaugh maps for the functions $f_1 = xw + yw + x'y'z$ and $f_2 = x'y + yw'$.
Hence derive the Karnaugh maps for $g = f_1 f_2$ and $h = f_1 + f_2$.
Simplify the maps for g and h, and give the resulting expressions in the sum-of-products form.

(b) Determine the state diagram and the state table for a single output circuit which detects a '01' sequence. The output $z = 1$, which is reset only by a '00' input sequence. For all other cases, $z = 0$. Design the circuit using JK flip-flops.

E8. (a) Consider the following circuit with an ideal Op-amp. Calculate V_o.



(b) The following network uses four transconductor amplifiers and two capacitors to produce the output voltage V_o for the input voltage V_i.

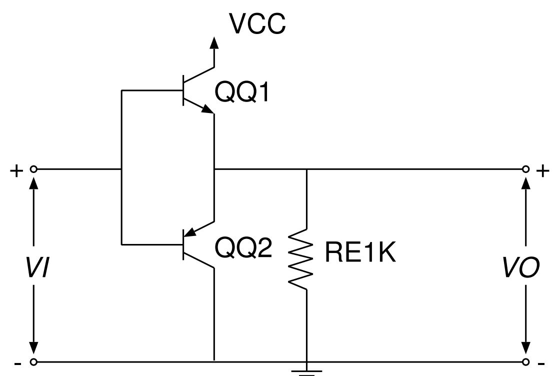


(i) Show that the voltage transfer function $H(s)$ can be expressed as:

$$H(s) = \frac{V_o}{V_i} = \frac{g_{m_1}/g_{m_4}}{1 + (\frac{g_{m_2}C_2}{g_{m_3}g_{m_4}})s + (\frac{C_1C_2}{g_{m_3}g_{m_4}})s^2}.$$

(ii) Does the transfer function suggest a lowpass, bandpass or highpass frequency response? Briefly explain.

E9. Consider the amplifier shown in the following figure.



(i) Draw the equivalent circuit using the small-signal hybrid parameter model.

(ii) For the following values of h parameters for both transistors: $h_{ie} = 1000\ \Omega$, $h_{fe} = 100$, $h_{re} = h_{oe} = 0$, determine the voltage amplification A_v and the input resistance R_{in}.

E10. The following is the skeleton of a C program that outputs the number of occurrences of each of the ten digits in a given string of digits. Write the codes for the portions marked as **B1**, **B2**, **B3** and **B4** with appropriate C constructs.

```
#include<stdio.h>
#define base 10
```

```
/* This program outputs the numbers of 0's, 1's */
/*  ,....., 9's  in an input string ending in $ */

int main() {
char b;
int i, a[base];

/* Initialize array elements to zero */
for (B1)
a[i] = 0;
printf("Input numeric characters ending with $\n");
scanf("%c", &b);  /* Scan next character */

/* Execute the loop as long as $ is not scanned */
while (B2) {
printf("Processing the digit %c\n",b);
B3; }
for (i=0; i<=9; i=i+1)
printf("No. of occurrences of %d is %d\n",i,a[i]);
B4;
printf("The most frequent digit is %d\n", i);
}
```

Mathematics

M1. (a) Evaluate

$$\lim_{k\to\infty}\int_0^\infty \frac{dx}{1+kx^{10}}\ .$$

(b) Is it possible to define $f : S \to T$ such that f is continuous and onto for each of the following pairs of S and T? For each pair, provide an example of one such f, if possible; otherwise, show that it is impossible to define one such f.

(i) $S = (0,1) \times (0,1)$ and T is the set of rational numbers.

(ii) $S = (0,1) \times (0,1)$ and $T = [0,1] \times [0,1]$.

M2. (a) Let B be a non-singular matrix. Then prove that λ is an eigenvalue of B if and only if $1/\lambda$ is an eigenvalue of B^{-1}.

(b) If rank(A) = rank(A^2) then show that

$$\{x\ :\ Ax = 0\}\ =\ \{x\ :\ A^2x = 0\}.$$

(c) Let

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Which of the following statements are true? In each case, justify your answer.

(i) The rank of A is equal to the trace of A.

(ii) The determinant of A is equal to the determinant of A^n for all $n > 1$.

M3. (a) (i) For $0 \le \theta \le \pi/2$, show that $\sin\theta \ge 2\theta/\pi$.

(ii) Hence or otherwise show that for $\lambda < 1$,

$$\lim_{x\to\infty} x^\lambda \int_0^{\pi/2} e^{-x\sin\theta} d\theta = 0.$$

(b) Let $a_n \ge 0, n = 1, 2, \ldots$ be such that $\sum a_n$ converges. Show that $\sum \sqrt{a_n} n^{-p}$ converges for every $p > 1/2$.

M4. (a) Let $a_1, a_2, \ldots$ be integers and suppose there exists an integer N such that $a_n = (n-1)$ for all $n \ge N$. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n!}$ is rational.

(b) Let $0 < s_1, s_2, s_3 < 1$. Show that there exists exactly one $x \in (0, \infty)$ such that

$$s_1^x + s_2^x + s_3^x = 1.$$

M5. (a) Let A be an $n \times n$ symmetric matrix and let $l_1, l_2, \ldots, l_{r+s}$ be $(r+s)$ linearly independent $n \times 1$ vectors such that for all $n \times 1$ vectors x,

$$x'Ax = (l_1'x)^2 + \cdots + (l_r'x)^2 - (l_{r+1}'x)^2 - \cdots - (l_{r+s}'x)^2.$$

Prove that $\text{rank}(A) = r + s$.

(b) Let A be an $m \times n$ matrix with $m < n$ and $\text{rank}(A) = m$. If $B = AA'$, $C = A'A$, and the eigenvalues and eigenvectors of B are known, find the non-zero eigenvalues and corresponding eigenvectors of C.

M6. (a) If T is an injective homomorphism of a finite dimensional vector space V onto a vector space W, prove that T maps a basis of V onto a basis of W.

(b) Find a polynomial of degree 4 which is irreducible over $GF(5)$. Justify your answer.

M7. (a) Let S and T be two subsets of a finite group $(G, +)$ such that $|S| + |T| > |G|$. Here $|X|$ is the number of elements in a set X. Then prove that

$$S + T = G, \quad \text{where} \quad S + T = \{s + t \ : \ s \in S, t \in T\}.$$

(b) A number x is a square modulo p if there is a y such that $y^2 \equiv x \bmod p$. Show that for an odd prime p, the number of squares modulo p is exactly $\frac{p+1}{2}$.

(c) Using (a), (b) or otherwise prove that for any integer n and any odd prime p, there exist x, y such that $n \equiv (x^2 + y^2) \bmod p$.

M8. (a) Give an example of a 3-regular graph on 16 vertices whose chromatic number is 4. Justify your answer.

(b) Give an example of a graph G such that both G and $\overline{G}$ are not planar. Justify your answer.

(c) A graph is said to be 2-connected if deleting any one vertex does not make the graph disconnected. Let G be a 2-connected graph.

(i) Suppose $e = (u, v)$ is an edge of G and x is a vertex of G where x is distinct from u and v. Show that there is a path from x to u which does not go through v.

(ii) Hence or otherwise, show that if e_1 and e_2 are two distinct edges of G, then they lie on a common cycle.

M9. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(x) = f\left(\frac{x}{1-x}\right) \text{ for all } x \neq 1.$$

Assuming that f is continuous at 0, find all possible such f.

(b) For any real-valued continuous function f on $\mathbb{R}$, show that

$$\int_0^x \left(\int_0^u f(t)dt\right) du = \int_0^x (x-u)f(u)du \text{ for } 0 < u < x.$$

M10 (i) $GL(2, \mathbb{Z}_2)$ denotes the group of 2×2 invertible matrices with entries in $\mathbb{Z}_2 = \{0, 1\}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The operation in $GL(2, \mathbb{Z}_2)$ is matrix multiplication with all the arithmetic done in $\mathbb{Z}_2$.

Is the cyclic subgroup generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ a normal subgroup? Justify your answer.

(ii) Consider the set $A = \{(0,0), (1,1), (2,2)\}$.

(i) Prove that A is a subring of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(ii) Prove or disprove: A is an ideal of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

M11 (i) Given a simple graph $G = (V, E)$ with $V = \{v_1, v_2, \cdots, v_n\}$ and $E = \{e_1, e_2, \cdots, e_m\}$, let $B = (b_{ij})_{n \times m}$ be the matrix such that

$$\begin{aligned} b_{ij} &= 1 \text{ if } v_i \in e_j \\ &= 0 \text{ otherwise .} \end{aligned}$$

Let A be the adjacency matrix of G, and D the diagonal matrix with the degree sequence $[d(v_1), d(v_2), \cdots, d(v_n)]$ on the diagonal. Show that $BB^T = A + D$.

(ii) Show that in a tree, there is a vertex common to all the longest paths in the tree.

M12 (a) Consider the differential equation:

$$e^x \sin y dx + e^x \cos y dy = y \sin(xy) dx + x \sin(xy) dy.$$

Find the equation of the particular curve that satisfies the above differential equation and passes through the point $\left(0, \frac{\pi}{2}\right)$.

(b) Let the function f be four times continuously differentiable on $[-1, 1]$ with $f^{(4)}(0) \neq 0$. For each $n \geq 1$, let

$$f\left(\frac{1}{n}\right) = f(0) + \frac{1}{n}f'(0) + \frac{1}{2n^2}f''(0) + \frac{1}{6n^3}f^{(3)}(\theta_n),$$

where $0 < \theta_n < \frac{1}{n}$.

Show that $n\theta_n \to \frac{1}{4}$ as $n \to \infty$.

Physics

P1. (a) Consider two partially overlapping spherical charge distributions with constant charge densities $+\rho$ and $-\rho$. Each sphere is of radius R. The vector connecting the center of the negative charge sphere to the center of the positive charge sphere is $\vec{D}$. Find the electrostatic field at any point in the overlapping region.

(b) Consider two wire loops L_1 and L_2. Show that the magnetic flux linked to L_1 when current I flows in L_2, is same as the magnetic flux linked to L_2 when current I flows in L_1.

(c) There are two co-axial solenoids. The inner short solenoid has radius R, length L, N_1 number of turns per unit length. The outer solenoid is very long with N_2 number of turns per unit length. Find the magnetic flux linked with the outer solenoid when current I flows in the inner solenoid. What is the coefficient of mutual inductance of the system of solenoids?
(Hint: You can use the answer of (b) in (c).)

P2. (a) A particle is falling freely from a height h at 30° latitude in the northern hemisphere. Show that the particle will undergo a deflection of $\omega\sqrt{\frac{2h^3}{3g}}$ in the eastward direction, where ω is the rotational velocity of the earth about its own axis and g is the acceleration due to gravity.

(b) A particle of mass m is moving in a plane in the field of force $\overrightarrow{F} = -\widehat{r}kr\cos\theta$, where k is a constant, $\widehat{r}$ is the radial unit vector and θ is the polar angle.

(i) Write the Lagrangian of the system.

(ii) Show that the Lagrange's equations of motion are:

A. $m\ddot{r} - mr\dot{\theta}^2 + kr\cos\theta = 0$;

B. $mr^2\dot{\theta} \neq$ constant.

(iii) Interpret (ii)B in the context of Kepler's second law.

P3. (a) (i) A photon of energy E_i is scattered by an electron of mass m_e that is initially at rest. The final energy of the photon is E_f. Let θ be the angle between the directions of the incident photon and the scattered photon. Using the principles of Special Theory of Relativity, find θ. (c is the velocity of light in vacuum.)

(ii) What is the minimum energy needed for a photon to produce an electron-positron pair if the photon collides with another particle?

(b) A free particle of mass m moves in one dimension. At time $t = 0$, the normalized wave function of the particle is

$$\psi(x,\ 0,\ \sigma_x^2) = (2\pi\sigma_x^2)^{-1/4}\exp(-x^2/4\sigma_x^2)$$

where $\sigma_x^2 = \langle x^2 \rangle$.

(i) Compute the momentum spread $\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$ associated with this wave function.

(ii) Show that at time $t > 0$ the probability density of the particle has the form $|\psi(x,t,\bar{\sigma}_x^2)|^2 = |\psi(x,t,\ \sigma_x^2 + \sigma_p^2 t^2/m^2)|^2$.

P4. (a) Calculate the following properties of the $2p - 1s$ electromagnetic transition in an atom formed by a muon and a strontium nucleus ($Z = 38$):

(i) the fine structure splitting energy;

(ii) the natural line-width (i.e., the part of the line-width of an absorption or emission line that results from the finite lifetimes of one or both of the energy levels between which the transition takes place).

Given: the lifetime of the $2p$ state of hydrogen is 10^{-9}s.

(b) Consider the following high energy reactions. Check whether the reactions are allowed or forbidden. If allowed, mention the corresponding decay process, and if forbidden, mention the law that is violated.

(i) $\mu^+ \to e^+ + \gamma$

(ii) $p + \bar{p} \to \gamma$

(iii) $p \to e^+ + \nu_e$

(iv) $p + n \to p + \Lambda^0$

(v) $p \to e^+ + n + \nu_e$

P5. (a) How does one understand molecular mean free path in the context of molecular kinetic theory of gases? Obtain the analytic form of the law governing the distribution of free paths in an ideal gas.

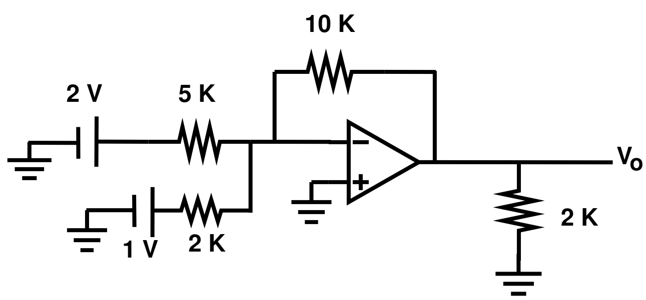
(b) Calculate the mean free path, the collision rate and the molecular diameter for Hydrogen gas molecules having the following particulars: molecular weight of Hydrogen = 2.016 gm; viscosity, $\eta = 85 \times 10^{-6}$ dynes/cm^2/velocity gradient; mean speed, $\overline{c} = 16 \times 10^4$ cm/sec; density, $\rho = 0.000089$ gm/cc.

P6. An electron is confined to move within a linear interval of length L. Assuming the potential to be zero throughout the interval except for the two end points, where the potential is infinite, find the following:

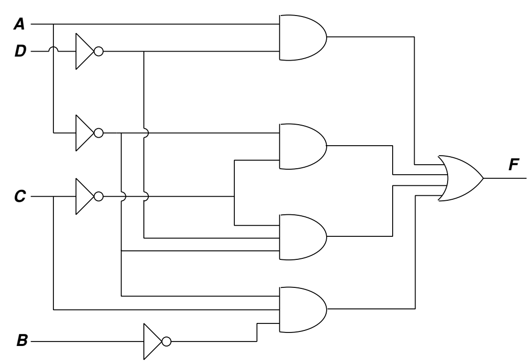
(a) What is the probability of finding the electron in the region $0 < x < \frac{L}{4}$, when it is in the lowest (ground) state of energy;

(b) Taking the mass of the electron me to be 9×10^{-31} Kg, Planck's constant h to be 6.6×10^{-34} Joule-sec and $L = 1.1$ cm, determine the electron's quantum number when it is in the state having an energy equal to 5×10^{-32} Joule.

P7. (a) Consider the following circuit with an ideal Op-amp. Calculate V_o.



(b) The circuit shown in the following figure computes a Boolean function F. Assuming that all gates cost Rs. 5 per input (i.e., an inverter costs Rs. 5, a 2-input gate costs Rs. 10, etc.), find the minimum cost realization of F using only inverters, AND / OR gates.



P8. (a) Lattice constant of a cubic lattice is a. Calculate the spacing between (2 1 1), (1 1 0), (1 1 1) and (1 0 1) planes.

(b) Show that for a crystal of cubic symmetry the direction [h, k, l] is perpendicular to the plane (h k l).

(c) The diamond crystal structure has the cube edge of 356Å. Calculate the distance between the nearest-neighbor atoms.

P9. The zero-voltage barrier height of an abrupt p-n junction is 0.35 volt. Assume that the concentration N_A of acceptor atoms on the p-side is much smaller than that of donor atoms on the n-side, and $N_A = \frac{10^{19}}{\pi}\text{m}^{-3}$. Calculate the width of the depletion layer for an applied reverse voltage of 14.65 volts.

If the cross-sectional area of the diode is 1 sq. mm, calculate the space charge capacitance corresponding to this applied reverse voltage. (Assume that the permittivity of the material is $12 \times 10^{-9}/36\pi$ farad/metre).

Statistics

S1. Let $p_1 > p_2 > 0$ and $p_1 + p_2 + p_3 = 1$. Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables where, for all i, $Pr[Y_i = j] = p_j$, $j = 1, 2, 3$. Let $S_{j,n}$ denote the number of Y_i's among $Y_1, \ldots, Y_n$ for which $Y_i = j$. Show that

$$\lim_{n\to\infty} Pr[S_{1,n} - S_{2,n} \ \geq\ 2] = 1.$$

S2. The random variables $X_1, X_2, \ldots, X_k$ are defined iteratively as follows:

X_1 is uniformly distributed on $\{1, \ldots, n\}$ and for $i \geq 2$, the distribution of X_i given $(X_1, \ldots, X_{i-1})$ is uniform on $\{1, 2, \ldots, X_{i-1}\}$.

Find $E(X_k)$ and compute $\lim\limits_{k\to\infty} E(X_k)$.

S3. Let X_1, X_2, X_3 and X_4 be independent random variables having a normal distribution with zero mean and unit variance. Show that

$$\frac{\sqrt{2}(X_1X_3 + X_2X_4)}{{X_3}^2 + {X_4}^2}$$

has a t distribution.

S4. (a) Let $X_1, \ldots, X_n$ be independent Poisson random variables with common expectation λ. Let $\widehat{\lambda} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$. Is $\exp(-\widehat{\lambda})$ an unbiased estimator of $\exp(-\lambda)$? Justify your answer.

(b) Let X_1, X_2 and X_3 be independent random variables such that X_i is uniformly distributed in $(0, i\theta)$ for $i = 1, 2, 3$. Find the maximum likelihood estimator of θ and examine whether it is unbiased for θ.

S5. Consider the following Gauss-Markov linear model:

$$\begin{aligned}
E(y_1) &= \theta_0 + \theta_1 + \theta_2, \\
E(y_2) &= \theta_0 + \theta_1 + \theta_3, \\
E(y_3) &= \theta_0 + \theta_2 + \theta_3, \\
E(y_4) &= \theta_0 + \theta_1 + \theta_2.
\end{aligned}$$

(a) Determine the condition under which the parametric function $\displaystyle\sum_{i=0}^{3} c_i\theta_i$ is estimable for known constants $c_i,\ i = 0, 1, 2, 3$.

(b) Obtain the least squares estimates of the parameters $\theta_0, \theta_1, \theta_2$ and θ_3.

(c) Obtain the best linear unbiased estimator of $(2\theta_1 - \theta_2 - \theta_3)$ and also determine its variance.

S12. Suppose Y is regressed on X_1, X_2 and X_3 with an intercept term and the following are computed:

$$Y'Y = 5000;\ Y'X = (20, 30, 50, -40);\ X'X = \begin{pmatrix} 20 & 0 & 0 & 0 \\ 0 & 19 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

(a) Compute the regression coefficients.

(b) Compute the ANOVA table.

(c) Compute the estimate of the error variance, and the estimates of the variances of all the regression coefficients.

S7. Suppose that a coin is tossed 10 times.

(a) Find the most powerful test at level $\alpha = 0.05$ to test whether the coin is fair against the alternative that the coin is more likely to show up heads.

(b) What will be the conclusion of the test if there are exactly 7 heads in 10 tosses?

(c) Find the power function of this test.

S8. (a) Consider a randomized block design with v treatments, each replicated r times. Let t_i be the effect of the i-th treatment. Find $Cov(\sum a_i\hat{t}_i, \sum b_i\hat{t}_i)$ where $\sum a_i\hat{t}_i$ and $\sum b_i\hat{t}_i$ are the best linear unbiased estimators of $\sum a_i t_i$ and $\sum b_i t_i$ respectively and $\sum a_i = \sum b_i = \sum a_i b_i = 0$.

(b) A sample S_1 of n units is selected from a population of N units using SRSWOR. Observations on a variable Y are obtained for the n_1 units of S_1 who responded. Later, a further sub-sample S_2 of m units is selected using SRSWOR out of the $(n - n_1)$ units of S_1 who did not respond. Assuming that Y could be observed for all the m units of S_2, find the following:

(i) an unbiased estimator of the population mean $\bar{Y}$ on the basis of the available observations on Y,

(ii) an expression for the variance of the proposed estimator.

S9. (a) Each time you buy a product, you get a coupon which can be any one of N different types of coupons. Assuming that the probability that a coupon of type i occurs is p_i, find the distribution of the random variable

X which denotes the total number of products to be bought in order to have all types of coupons.

(b) A box contains $6n$ tickets numbered $0, 1, 2, \ldots, 6n-1$. Three tickets are drawn at random without replacement. Find the probability that the sum of the three numbers selected is $6n$.

S10. Let

$$p(x;\theta,k) = \frac{\theta^k}{\Gamma(k)}\, e^{-\theta x} x^{k-1}, \text{ where } 0 < x < \infty, \theta > 0 \text{ and } k > 0.$$

Find minimum variance unbiased estimate of $\frac{1}{\theta}$.

S11. (a) Suppose in a coin tossing experiment with $2n$ trials, an unbiased coin is flipped n times, while a different (possibly biased) spurious coin is flipped remaining n times by mistake. The total number of *heads* is found to be S_{2n} in the $2n$ trials.

(i) Based on S_{2n}, describe a test for the hypothesis that the spurious coin is actually unbiased.

(ii) Give an approximate cut-off point at $\alpha = 0.05$, assuming n is large.

(b) Let $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ be independent observations from populations with continuous distribution functions F_1 and F_2. Denote by m_1 and n_1 the number of x's and y's exceeding the *k*th order statistic of the combined sample.

Derive a nonparametric test of the null hypothesis $H_0 : F_1 = F_2$, based on the propability distribution of (m_1, n_1) under H_0.

S12. Consider a distribution of shots fired at a target point. Let (X, Y) be the coordinates *(random variables)* representing the errors of a shot with respect to the two orthogonal axes through the target point.

Let the following hypotheses be true:

I. The marginal density functions $p(x)$, $q(y)$ of the errors X and Y are continuous.

II. The probability density at (x, y) depends only on the distance $r = (x^2 + y^2)^{1/2}$ from the target point.

III. X and Y are independent.

Show that X and Y are identically distributed, and the probability distribution function of X is

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2} \quad \text{for some } \sigma > 0.$$

—x—